Kingdom of Saudi Arabia

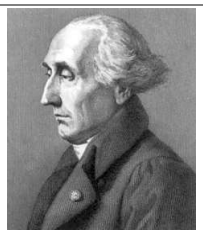
Al-Imam Muhammad Ibn Saud Islamic University

College of Sciences- Department of Physics

PHYS. 303

Isaac Newton
1643 - 1727

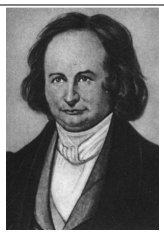
Leonhard Euler
1707 - 1783


Joseph L. Lagrange
1736 - 1813

CLASSICAL

MECHANICS

II


Carl Jacobi
1804 - 1851

William Hamilton
1805 - 1865

Albert Einstein
1879 - 1955

Prof. Dr. Ali S. Hennache

A.S.H. /AIMISIU/June 2012
Riyadh- KSA

KINGDOM OF SAUDI ARABIA
AL-IMAM MUHAMMAD BIN SAUD
ISLAMIC UNIVERSITY
FACULTY OF SCIENCES
DEPARTMENT OF PHYSICS

المملكة العربية السعودية
الإسلامية جامعة الإمام محمد بن سعود
كلية العلوم
قسم الفيزياء

Level Six

Classical Mechanics (2)

Course Code	Course Num.	Course Name	Credit Hours	Lec	Lab	Tut	Prerequisites
PHY	303	Classical Mechanics (2)	3	3	0	1	PHY 105, MAT 203

Objectives:

At the end of the unit the student should:

- understand the notions of configuration space, generalized coordinates space in mechanics
- be able to obtain the Euler-Lagrange equations from a variational principle
- understand the relation between Lagrange's equations and Newton's laws
- be able to use Lagrange's equations to solve complex mechanical problems
- be able to obtain the Hamiltonian formulation of a mechanical system

Syllabus:

- Some Methods in the Calculus of Variations: Euler's equation, the second form of the Euler equation, functions with several dependent variables, Euler's equation when auxiliary conditions are imposed.
- Lagrangian and Hamiltonian Mechanics: Hamiltonian's principle, generalized coordinates, Lagrange's equations of motion in generalized coordinates, Lagrange's equations with undetermined multipliers, equivalence of Lagrange's and Newton's equations, a theorem concerning the kinetic energy, conservation theorems, canonical equations of motion – Hamiltonian mechanics.
- Central Force Motion: reduced mass, conservation theorems-first integrals of the motion, planetary motion-Kepler's problem.
- Mechanics of a System of Particles: center of mass, linear momentum of the system-energy of the system, elastic collisions, inelastic collisions.
- Motion in a noninertial reference frame: rotating coordinate systems, centrifugal and Coriolis forces, motion relative to the earth.
- Mechanics of rigid Bodies: inertia tensor, angular momentum, principal axes of inertia, moments of inertia for different body coordinate systems, Eulerian angles, Euler's equations for a rigid body.
- Introduction to Coupled Oscillations: two coupled harmonic oscillators, weak coupling, general problem of coupled oscillations.

References:

Text book: http://teacher.pas.rochester.edu/PHY235/LectureNotes/

- Classical Dynamics of Particles and Systems, Jerry Marion and Stephen T., Brookes Cole, 5 edition (2003)
- Mechanics, L. Landau and E. Lifshitz, third edition. Butterworth-Heinmann, Oxford, 1976
- Introduction to Classical Mechanics 2nd Ed., Atam P. Arya, Prentice Hall, New Jersey (0-13-505223-8)
- Analytical Mechanics 5th Ed., Grant R. Fowles and George L. Cassidy, Saunders, New York (0-03-096022-3)

Kingdom of Saudi Arabia

Al-Imam Muhammad Ibn Saud Islamic University

Faculty of Sciences

Department of Physics

GRADING POLICY AND EXAMS SCHEDULE

Prof. Dr. Ali S. Hennache

EXAMS SCHEDULE:

	COURSE (SUBJECT)	CODE	QUIZ 1	QUIZ 2	MIDTERM 1	MIDTERM 2
1	CLASSICAL MECHANICS II	PHYS 303	02 July 2012	04 July 2011	09 July 2012	17 July 2012

GRADING:

PHYS 303:

Grading for PHYS 303 will be based on an absolute scale of 100 points, distributed according to the following table:

Exam/Assignment	Number of Exams / Assignments	Possible Points Each:	Total Possible Points:
Final Exam	1	40	40
Midterm Exam	2	20	40
Quiz	2	05	10
Attendance/Participation/Assignments			10
Lab	00	00	00
TOTAL POINTS			100

A.S.H/AIMISIU/CS/DP/20.06.2012-Riyadh

Some Methods in the Calculus of Variations

In this Chapter we focus on an important method of solving certain problems in Classical Mechanics. In many problems we need to determine how a system evolves between an initial state and a final state. For example, consider two locations on a two-dimensional plane. Consider an object moving from one location to the other, and assume the object experiences a friction force when it moves over the surface. One can ask questions such as "What is the path between the initial and the final condition that minimizes the work done by the friction force?" that can be most easily answered using the calculus of variations. The evolution that can be studied using the calculus of variations is not limited to evolutions in real space. For example, one can consider the evolution of a gas in a pV diagram and ask "What is the path between the initial and final state that maximizes the work by the gas?"

Euler's Equation

Consider the two-dimensional plane shown in Figure 1. Our initial position is specified by (x_1,y_1) and the final position is specified by (x_2,y_2).

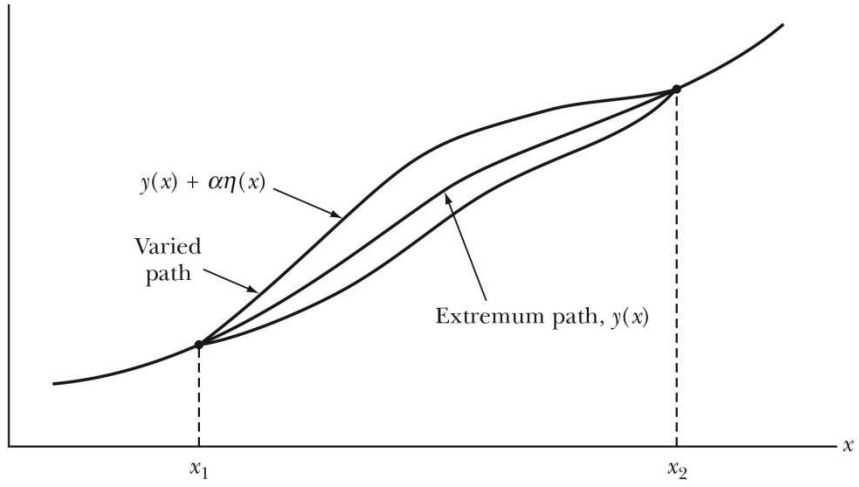


Figure 1. Path $y(x)$ that is used to move from position 1 to position 2.

Consider that we are asked to minimize the path integral of a function f between position 1 and position 2. Suppose the path integral is minimized when the use path $y(x)$. If we change the path slightly by adding a second function $n(x)$ then we expect that the path integral is expected to increase. Consider the following path:

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

The function $\eta(x)$ is a arbitrary function of x and is used to make small changes to the path. The only requirements of $\eta(x)$ are that $\eta(x)$ has a continuous first derivative and that $\eta(x)$ vanishes at the end points of the path, that is $\eta(x_1) = \eta(x_2) = 0$.

Since the path integral is minimized when we follow the path, the path integral must have an extreme value when $\alpha = 0$. This requires that

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\left(y(\alpha,x), y'(\alpha,x), x\right)dx\Bigg|_{\alpha=0} = 0$$

The left hand side of this equation can be rewritten by differentiating the argument of the integral with respect to α:

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)dx\Bigg|_{\alpha=0} = 0$$

Using our definition of $y(\alpha, x)$ we can easily show that

$$\frac{\partial y}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(y(0,x) + \alpha\eta(x)\right) = \eta(x)$$

and

$$\frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\frac{dy}{dx} = \frac{\partial}{\partial \alpha}\left(\frac{dy(0,x)}{dx} + \alpha\frac{d\eta(x)}{dx}\right) = \frac{d\eta(x)}{dx}$$

The requirement that the path integral has an extreme is equivalent to requiring that

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx}\right)dx\Bigg|_{\alpha=0} = 0$$

The second term in the integrant can be rewritten as

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx}\right)dx\Bigg|_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y'}d\eta(x)\right)\Bigg|_{\alpha=0} = \frac{\partial f}{\partial y'}\eta(x)\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\right)dx$$

This path integral is an extreme if

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x)-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\right)dx\Bigg|_{\alpha=0}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx\Bigg|_{\alpha=0}=0$$

Since $\eta(x)$ is an arbitrary function, this equation can only satisfied if

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

This equation is known as **Euler's equation**.

Example: Problem 6.4

Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

The element of distance along the surface of a cylinder is

$$dS=\sqrt{(dx)^2+(dy)^2+(dz)^2} \tag{6.4.1}$$

In cylindrical coordinates (x, y, z) are related to (ρ, ϕ, z) by

$$\left.\begin{aligned} x&=\rho\cos\phi\\ y&=\rho\sin\phi\\ z&=z \end{aligned}\right] \tag{6.4.2}$$

Since we consider motion on the surface of a cylinder, the radius ρ is constant. The expression for x, y, and z can be used to express dx, dy, and dz in cylindrical coordinates:

$$\left.\begin{aligned} dx&=-\rho\sin\phi\,d\phi\\ dy&=\rho\cos\phi\,d\phi\\ dz&=dz \end{aligned}\right] \tag{6.4.3}$$

Substituting (6.4.3) into (6.4.1) and integrating along the entire path, we find

$$S = \int_1^2 \sqrt{\rho^2 (d\phi)^2 + (dz)^2} = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + \dot{z}^2}\, d\phi \tag{6.4.4}$$

If S is to be a minimum, $f = \sqrt{\rho^2 + \dot{z}^2}$ must satisfy the Euler equation:

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial \phi}\frac{\partial f}{\partial \dot{z}} = 0 \tag{6.4.5}$$

Since $\frac{\partial f}{\partial z} = 0$, the Euler equation becomes

$$\frac{\partial}{\partial \phi}\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = 0 \tag{6.4.6}$$

This condition will be satisfied if

$$\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = \text{constant} \equiv C \tag{6.4.7}$$

or,

$$\dot{z} = \sqrt{\frac{C^2}{1 - C^2}}\,\rho \tag{6.4.8}$$

Since ρ is constant, (6.4.8) implies that

$$\frac{dz}{d\phi} = \text{constant}$$

and for any point along the path, z and ϕ change at the same rate. The curve described by this condition is a *helix*.

Second Form of Euler's Equation

In some applications, the function f may not depend explicitly on x: $\partial f/\partial x = 0$. In order to benefit from this constraint, it would be good to try to rewrite Euler's equation with a term $\partial f/\partial x$ instead of a term $\partial f/\partial y$.

The first step in this process is to examine df/dx. The general expression for df/dx is

$$\frac{df}{dx} = \frac{d}{dx} f(y, y' : x) = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}$$

Note that we have not assume that $\partial f/\partial x = 0$ in order to derive this expression for df/dx. This equation can be rewritten as

$$y''\frac{\partial f}{\partial y'} = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y}$$

We also know that

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} = \left(\frac{df}{dx} - y'\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) + y'\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{df}{dx} - \frac{\partial f}{\partial x} + y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)$$

Applying Euler's theorem to the term in the parenthesis on the right hand side, we can rewrite this equation as

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x}$$

or

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

This equation is called **the second form of Euler's equation**. If f does not depend explicitly on x we conclude that

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

Example: Problem 6.4 – Part II

Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

We have already solved this problem using the "normal" form of Euler's equation. However, looking back at the solution we realize that the expression for *f*, $f = \sqrt{\rho^2 + \dot{z}^2}$, does not depend explicitly on ϕ. We thus should be able to use the second form of Euler's equation to solve this problem.

$$f - \dot{z}\frac{\partial f}{\partial \dot{z}} = \sqrt{\rho^2 + \dot{z}^2} - \dot{z}\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = \frac{\rho^2}{\sqrt{\rho^2 + \dot{z}^2}} = \text{constant} = C$$

This equation requires that

$$\frac{dz}{d\phi} = \sqrt{\frac{\rho^4}{C^2} - \rho^2} = \rho\sqrt{\frac{\rho^2}{C^2} - 1} = \text{constant}$$

Since ρ is constant, this equation implies that

$$\frac{dz}{d\phi} = \text{constant}$$

and for any point along the path, z and ϕ change at the same rate. The curve described by this condition is a *helix*.

Euler's Equation with Several Dependent Variables

Consider a situation where the function *f* depends on several dependent variables $y_1, y_2, y_3, \ldots$ etc., each of which depends on the independent variable x. Each dependent variable $y_i(\alpha, x)$ is related to the solution $y_i(0, x)$ in the following manner:

$$y_i(\alpha, x) = y_i(\alpha, x) + \alpha\eta_i(x)$$

If the independent functions y_1, y_2, y_3, etc. minimize the path integral of *f*, they must satisfy the following condition:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

for $i = 1, 2, 3, \ldots$. The procedure to find the optimum paths is similar to the procedures we have discussed already, except that we need to solve the Euler equation for each dependent variable y_i.

Euler's Equation with Boundary Conditions

In many cases, the dependent variable y must satisfy certain boundary conditions. For example, in problem 6.4 the function y must be located on the surface of the cylinder. In this case, any point on y must satisfy the following condition:

$$r = \rho = \text{constant}$$

In general, we can specify the constraint on the path(s) by using one or more functions g and requiring that $g\{y_i; x\} = 0$.

Let's start with the case where we have two dependent variables y and z. In this case, we can write the function f as

$$f = f\{y, y', z, z'; x\}$$

In this case, we can write the differential of J with respect to α as

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) \eta_y(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \eta_z(x) \right\} dx$$

Using our definition of $y_i(\alpha, x)$ we can rewrite this relation as

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \frac{\partial z}{\partial \alpha} \right\} dx$$

For our choice of constraint we can immediately see that the derivative of g must be zero:

$$\frac{dg}{d\alpha} = \left(\frac{\partial g}{\partial y} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial g}{\partial z} \right) \frac{\partial z}{\partial \alpha} = 0$$

Using our definition of y and z we can rewrite this equation as

$$\frac{dg}{d\alpha} = \left(\frac{\partial g}{\partial y} \right) \eta_y + \left(\frac{\partial g}{\partial z} \right) \eta_z = 0$$

This equation shows us a general relation between the functions η_x and η_y:

$$\frac{\eta_z}{\eta_y} = -\frac{\left(\frac{\partial g}{\partial y} \right)}{\left(\frac{\partial g}{\partial z} \right)}$$

Using this relation we can rewrite our expression for the differential of J:

$$\frac{\partial J}{\partial \alpha} = \int_{x_0}^{x_2} \left\{ \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) \eta_y(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \eta_z(x) \right\} dx =$$

$$= \int_{x_0}^{x_2} \left\{ \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \frac{\eta_z(x)}{\eta_y(x)} \right\} \{ \eta_y(x) \} dx$$

$$= \int_{x_0}^{x_2} \left\{ \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \frac{\left(\frac{\partial g}{\partial y} \right)}{\left(\frac{\partial g}{\partial z} \right)} \right\} \{ \eta_y(x) \} dx$$

Since the function η_x is an arbitrary function, the equation can only evaluate to 0 if the term in the brackets is equal to 0:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \frac{\left(\frac{\partial g}{\partial y} \right)}{\left(\frac{\partial g}{\partial z} \right)} = 0$$

This equation can be rewritten as

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \left(\frac{\partial g}{\partial z} \right)^{-1}$$

This equation can only be correct if both sides are equal to a function that depends only on x:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'} \right) \right) \left(\frac{\partial g}{\partial z} \right)^{-1} = -\lambda(x)$$

or

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right) + \lambda(x) \left(\frac{\partial g}{\partial y} \right) = 0$$

$$\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)+\lambda(x)\left(\frac{\partial g}{\partial z}\right)=0$$

Note that in this case, where we have one auxiliary condition, $g\{y, z; x\} = 0$, we end up with one Lagrange undetermined multiplier $\lambda(x)$. Since we have three equations and three unknown, y, z, and λ, we can determine the unknown.

In certain problems the constraint can only be written in integral form. For example, the constraint for problems dealing with ropes will be that the total length of the path is equal to the length of the rope L:

$$K[y]=\int g\{y,y';x\}dx=L$$

The curve y then must satisfy the following differential equation:

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)+\lambda(x)\left(\frac{\partial g}{\partial y}-\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right)=0$$

The δ notation

It is common to use the δ notation in the calculus of variations. In order to use the δ notation we use the following definitions:

$$\delta y=\frac{\partial y}{\partial \alpha}d\alpha$$

$$\delta J=\frac{\partial J}{\partial \alpha}d\alpha$$

In terms of these variables we find

$$\delta J=\frac{\partial J}{\partial \alpha}d\alpha = \left\{\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\frac{\partial y}{\partial \alpha}dx\right\}d\alpha=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\frac{\partial y}{\partial \alpha}d\alpha dx=$$
$$=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\delta y dx$$

Since δy is an arbitrary function, the requirement that $\delta J = 0$ requires that the term in the parenthesis is 0. This of course is the Euler equation we have encountered before!

It is important to not that there is a significant difference between δy and dy. Based on the definition of δy we see that δy tells us how y varies when we change α while keeping all other variables fixed (including for example the time t).

Hamilton's Principle - Lagrangian and Hamiltonian Dynamics

Many interesting physics systems describe systems of particles on which many forces are acting. Some of these forces are immediately obvious to the person studying the system since they are externally applied. Other forces are not immediately obvious, and are applied by the external constraints imposed on the system. These forces are often difficult to quantify, but the effect of these forces is easy to describe. Trying to describe such a system in terms of Newton's equations of motion is often difficult since it requires us to specify the total force. In this Chapter we will see that describing such a system by applying Hamilton's principle will allow us to determine the equation of motion for system for which we would not be able to derive these equations easily on the basis of Newton's laws. We should stress however, that Hamilton's principle does not provide us with a new physical theory, but it allows us to describe the existing theories in a new and elegant framework.

Hamilton's Principle

The evolution of many physical systems involves the minimization of certain physical quantities. We already have encountered an example of such a system, namely the case of refraction where light will propagate in such a way that the total time of flight is minimized. This same principle can be used to explain the law of reflection: the angle of incidence is equal to the angle of reflection.

The minimization approach to physics was formalized in detail by Hamilton, and resulted in Hamilton's Principle which states:

> " *Of all the possible paths along which a dynamical system may more from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* "

We can express this principle in terms of the calculus of variations:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0$$

The quantity T - U is called the Lagrangian L.

Consider first a single particle, moving in a conservative force field. For such a particle, the kinetic energy T will just be a function of the velocity of the particle, and the potential energy will just be a function of the position of the particle. The Lagrangian is thus also a function of the position and the velocity of the particle. Hamilton's theorem states that we need to minimize the Lagrangian and thus require that

$$\delta\int_{t_1}^{t_2} L(x_i,\dot{x}_i)dt = 0$$

In Chapter 6 we have developed the theory required to solve problems of this type and found that the Lagrangian must satisfy the following relation:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

These last equations are called the **Lagrange equations of motion.** Note that in order to generate these equations of motion, we do not need to know the forces. Information about the forces is included in the details of the kinetic and potential energy of the system.

Consider the example of a plane pendulum. For this system, there is only one coordinate we need to specify, namely the polar angle θ. The kinetic energy T of the pendulum is equal to

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

and the potential energy U is given by

$$U = mgl(1-\cos\theta)$$

The Lagrangian for this system is thus equal to

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

The equation of motion can now be determined and is found to be equal to

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta - ml^2\ddot{\theta} = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This equation is of course the same equation we can find by applying Newton's force laws. I this example, the only coordinate that was used was the polar angle θ. Even though the pendulum is a 3-dimensional system, the constraints imposed upon its motion reduced the number of degrees of freedom from 3 to 1.

Generalized Coordinates

If we try to describe a system of n particles, we need in general $3n$ coordinates to specify the position of its components. If external constraints are imposed on the system, the number of degrees of freedom may be less. If there are m constraints applied, the number of degrees of freedom will be $3n - m$. The coordinates do not need to be the coordinates of a coordinate system, but can be any set of quantities that completely specifies the state of the system. The state of the system is thus full specified by a point in the configuration space (which is a $3n - m$ dimensional space). The time evolution of the system can be described by a path in the configuration space.

The generalized coordinates of a system are written as q_1, q_2, q_3, …… The generalized coordinates are of course related to the physical coordinates of the particles being described:

$$x_{\alpha,i} = x_{\alpha,i}\left(q_1, q_2, q_3, \ldots, t\right) = x_{\alpha,i}\left(q_j, t\right)$$

where $i = 1, 2, 3$ and $\alpha = 1, 2, \ldots, n$. Since the generalized coordinates in general will depend on time, we can also introduce the generalized velocities. The physical velocities will depends on the generalized velocities:

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}\left(q_j, \dot{q}_j, t\right)$$

Equations of Motion in Generalized Coordinates

Based on the introduction of the Lagrangian and generalized coordinates, we can rephrase Hamilton's principle in the following way:

> " *Of all the possible paths along which a dynamical system may more from one point to another in configuration space within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.* "

Thus

$$\delta \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i, t\right) dt = 0$$

and

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

When we use the Lagrange's equations to describe the evolution of a system, we must recognize that these equations are only correct of the following conditions are met:

1. 1. the force acting on the system, except the forces of constraint, must be derivable from one or more potentials.
2. 2. the equations of constraint must be relations that connect the coordinates of the particles, and may be time dependent (note: this means that they are independent of velocity).

Constraints that do not depend on velocity are called **holonomic constraints**. There are two different types of holonomic constraints:

1. 1. fixed or **scleronomic constraints**: constraints that do not depend on time.
2. 2. moving or **rheonomic constraints**: constraints that depend on time.

Lagrange's Equations with Undetermined Multipliers

We have seen already a number of examples were one could remove the equations of constraint by a suitable choice of coordinates. For example, when we looked at the motion of an object on the surface of a cylinder we could either:

1. 1. Use a set of three coordinates to describe the motion, coupled with one equation of constraint.
2. 2. Use a set of two coordinates (such at the azimuthal angle and the vertical position) to describe the motion, without an equation of constraint.

In this Section we will look at situations where the constraint depends on the velocity:

$$f\left(x_{\alpha,i},\dot{x}_{\alpha,i},t\right)=0$$

If the constraints can be expressed in a differential form,

$$\sum_{j=1}^{n}\frac{\partial f_k}{\partial q_j}dq_j=0$$

we can directly incorporate it into the Lagrange equations:

$$\frac{\partial L}{\partial q_j}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j}+\sum_{k=1}^{m}\lambda_k(t)\frac{\partial f_k}{\partial q_j}=0$$

It turns out that the forces of constraint can be determined from the equations of constraints and the Lagrange multipliers $\lambda_m(t)$:

$$Q_j=\sum_{k=1}^{m}\lambda_k(t)\frac{\partial f_k}{\partial q_j}$$

where Q_j is the j^{th} component of the generalized force, expressed in generalized coordinates.

The use of Lagrange multiplier to determine the forces of constraints is nicely illustrated in Example 7.9 in the textbook, where a disk rolling down an inclined plane is being studied. If the disk does not slip, we find that the distance along the plane y and the angle of rotation θ are related, and the equation of constraint is

$$f(y,\theta) = y - R\theta = 0$$

The textbook explains in detail how the Lagrange equations are solved in this case, and I will not reproduce this here. The solution shows us that the Lagrange multiplier is given by

$$\lambda = -\frac{Mg\sin\alpha}{3}$$

By combining the equation of constraint and the Lagrange multipliers we can determine the generalized forces of constraint:

$$Q_y = \lambda(t)\frac{\partial f}{\partial y} = -\frac{Mg\sin\alpha}{3}$$

and

$$Q_\theta = \lambda(t)\frac{\partial f}{\partial \theta} = \frac{MgR\sin\alpha}{3}$$

Note that these forces of constraint do not have to be all pure forces. The force of constraint associated with y is the friction force between the disk and the plane that is required to ensure that the disk rolls without slipping. However, the force of constraint associated with the angle θ is the torque of this friction force with respect to the center of the disk. We need to note the generalized force does not have to have the unit of force.

It is also important to note that if we had chosen to solve the problem by expressing the Lagrangian in terms of a single coordinate y, by eliminating the angle, we would not have obtained any information about the forces of constraint. Although I have stressed that in many cases, you can simplify the solution by the proper choice of coordinates such that the equations of constraint are eliminated, in this case, the solution will not provide any information about the forces of constraint.

The generalized momentum

One of the big differences between the equations of motion obtained from the Lagrange equations and those obtained from Newton's equations is that in the latter case, the coordinate

frame used is always a Cartesian coordinate frame. When we use the Lagrange equations we have the option to choice generalized coordinates that do not have to correspond to the coordinates of a Cartesian coordinate system.

The generalized coordinates are related to the Cartesian coordinates, and transformation rules allow use to carry out transformations between coordinate systems. The generalized forces of constraint are related to the Newtonian forces of constraint, as was illustrated in Example 7.9 in the textbook. The similarities between the Cartesian and the generalized parameters suggest it may also be useful to consider the concept of the generalized momentum.

In a Cartesian coordinate system we can easily determine the connection between the Lagrangian and the linear momentum. The Lagrangian is equal to

$$L = T - U = \frac{1}{2}m\sum_{i=1}^{3}\dot{x}_i^2 - U(x_i)$$

The Lagrange equation for this Lagrangian is given by

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = 0$$

and we can rewrite this as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} = F_i = m\dot{x}_i = \frac{d}{dt}p_i$$

This last equation suggest that we define the generalized momentum of a particle in the following way:

$$p_i = \frac{\partial T}{\partial \dot{x}_i}$$

It is obviously consistent with our definition of linear momentum in Cartesian coordinates.

Consider a particle moving in a two-dimensional plane and having its motion described in terms of spherical coordinates. The kinetic energy of the particle is equal to

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Since there are two generalized coordinates we can determine two generalized momenta:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

which is the linear momentum of the particle, and

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

which is the angular momentum of the particle. We thus see that two distinct concepts from our introductory courses emerge directly from our Lagrangian theory.

Homogeneous functions

Consider a homogeneous quadratic function f that depends only on the products of the generalized velocities:

$$f = \sum_{j,k} a_{j,k}\dot{q}_j\dot{q}_k$$

An example of such function would be the kinetic energy of a particle. Consider what happens when we differentiate this function with respect to one of the generalized velocities:

$$\frac{\partial f}{\partial \dot{q}_l} = \sum_j a_{j,l}\dot{q}_j + \sum_k a_{l,k}\dot{q}_k$$

If we multiply this equation by dq_l/dt and sum over all values of l we obtain:

$$\sum_l \dot{q}_l \frac{\partial f}{\partial \dot{q}_l} = \sum_l \dot{q}_l \left(\sum_j a_{j,l}\dot{q}_j + \sum_k a_{l,k}\dot{q}_k \right) = \sum_{k,j} a_{j,l}\dot{q}_j\dot{q}_l + \sum_{l,k} a_{l,k}\dot{q}_k\dot{q}_l = 2\sum_{j,k} a_{j,k}\dot{q}_j\dot{q}_k = 2f$$

In general, if f is a homogeneous function of the parameter y_k^n, then

$$\sum_k q_k \frac{\partial f}{\partial q_k} = nf$$

Conservation of Energy

If we consider a closed system, a system that does not interact with its environment, then we expect that the Lagrangian that describes this system does not depend explicitly on time. That is

$$\frac{\partial L}{\partial t} = 0$$

Of course, this does not mean that $dL/dt = 0$ since

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_j + \frac{\partial L}{\partial t} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_i$$

Using the Lagrange equations, we can rewrite this equation as

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_j = \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

This equation can be written as

$$\frac{d}{dt}\left(L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)\right) = 0$$

or

$$L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \text{constant} = -H$$

The constant H is called the Hamiltonian of the system and the Hamiltonian is defined as

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L$$

The Hamiltonian H is a conserved quantity for the system we are currently considering. If we use Cartesian coordinates we find that

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L = 2T - (T - U) = (T + U) = E$$

In this case we find that the Hamiltonian of the system is equal to the total energy of the system, and we thus conclude that the total energy is conserved. The equality of H and E is only satisfied if the following conditions are met:

- • The potential U depends only on position, and not on velocity.
- • The transformation rules connecting Cartesian and generalized coordinates are independent of time.

The latter condition is not met in for example a moving coordinate system, and in such system, the Hamiltonian will not be equal to the total energy.

We thus conclude that if the Lagrangian of a system does not depend explicitly on time, the total energy of that system will be conserved.

Conservation of Linear Momentum

The Lagrangian should be unaffected by a translation of the entire system in space, assuming that space is homogeneous (which is one of the requirement of an inertial reference frame). Consider what happens when we carry out an infinitesimal displacement of the coordinate system along one of the coordinate axes. The change in the Lagrangian as a result of this displacement must be equal to zero:

$$\delta L = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

We can rewrite this equation as

$$\delta L = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\left(\frac{d}{dt}\delta x_i\right) = \frac{\partial L}{\partial x_i}\delta x_i = 0$$

Since the displacement is arbitrary, this equation can only be correct if

$$\frac{\partial L}{\partial x_i} = 0$$

Using the Lagrange equation this is equivalent to requiring

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

or

$$\frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Assuming that the potential U does not depend on velocity we see that this relation is equivalent to

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = p_i = \text{constant}$$

The consequence of the independence of the Lagrangian under a translation of space is that linear momentum is conserved.

Conservation of Angular Momentum

Space is an inertial reference frame is isotropic, which means that the properties of a system are unaffected by the orientation of the system. In this case we expect that the Lagrangian does not change when the coordinate axes are rotated through an infinitesimal angle. A rotation through such an angle produces the following change in the position vector:

$$\delta\bar{r} = \delta\bar{\theta} \times \bar{r}$$

The velocity vectors will change in the same way:

$$\delta\dot{\bar{r}} = \delta\bar{\theta} \times \dot{\bar{r}}$$

The Lagrangian should not change as a result of such a transformation. Thus we must require that

$$\delta L = \sum_i \left(\frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right) = \sum_i \left\{ \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right\} = \sum_i \left(\dot{p}_i \delta x_i + p_i \delta \dot{x}_i \right) = 0$$

We thus conclude that

$$\dot{\bar{p}} \cdot \delta\bar{r} + \bar{p} \cdot \delta\dot{\bar{r}} = 0$$

When we express the changes in terms of the rotation angle we obtain:

$$\dot{\bar{p}} \cdot \left(\delta\bar{\theta} \times \bar{r} \right) + \bar{p} \cdot \left(\delta\bar{\theta} \times \dot{\bar{r}} \right) = \delta\bar{\theta} \cdot \left[\dot{\bar{r}} \times \bar{p} + \bar{r} \times \dot{\bar{p}} \right] = \delta\bar{\theta} \cdot \left[\frac{d}{dt} \left(\bar{r} \times \bar{p} \right) \right] = 0$$

Since the angle of rotation was an arbitrary angle, this relation can only be satisfied if

$$\frac{d}{dt} \left(\bar{r} \times \bar{p} \right) = 0$$

or

$$\bar{r} \times \bar{p} = \bar{L} = \text{constant}$$

The angular momentum of the system is thus conserved. This conserved quantity is a direct consequence of the invariance of the Lagrangian for infinitesimal rotations. We conclude that the important conserved quantities are a direct consequence of the properties of the space (and its symmetries).

Canonical Equations of Motion

The Lagrangian we have discussed in this Chapter is a function of the generalized position and the generalized velocity. The equations of motion can also be expressed in terms of the generalized position and the generalized momentum. The generalized momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

We can use the generalized momentum to rewrite the Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

The Hamiltonian can also be expressed in terms of the generalized momentum

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum_j \dot{q}_j p_j - L$$

In general we will write the Hamiltonian in terms of the generalized position and the generalized momentum . The change in H due to small changes in time and in the generalized position and momentum is equal to

$$\begin{aligned} dH &= \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt = \\ &= \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - p_j d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt = \\ &= \sum_j \left(\dot{q}_j dp_j - \frac{\partial L}{\partial q_j} dq_j \right) - \frac{\partial L}{\partial t} dt = \sum_j \left(\dot{q}_j dp_j - \dot{p}_j dq_j \right) - \frac{\partial L}{\partial t} dt \end{aligned}$$

The change in H can also be expressed in the following way:

$$dH = \sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt$$

After combining the last two equations we obtained the following relation:

$$\sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt = \sum_j \left(\dot{q}_j dp_j - \dot{p}_j dq_j \right) - \frac{\partial L}{\partial t} dt$$

or

$$\sum_i \left(\left\{ \frac{\partial H}{\partial p_i} - \dot{q}_i \right\} dp_i + \left\{ \frac{\partial H}{\partial q_i} + \dot{p}_i \right\} dq_i \right) + \left\{ \frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} \right\} dt = 0$$

Since the variations in time and the generalized position and momenta are equal to independent, the coefficients of dq_i, dp_i, and dt must be zero. Thus:

$$\frac{\partial H}{\partial p_i} - \dot{q}_i = 0$$
$$\frac{\partial H}{\partial q_i} + \dot{p}_i = 0$$
$$\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} = 0$$

The first two equations are called **Hamilton's equations of motion** or the **canonical equations of motion**. Note:

- For each generalized coordinate there are two canonical equations of motion.
- For each generalized coordinate these is only one Lagrange equations of motion.
- Each canonical equation of motion is a first order differential equation.
- Each Lagrange equation of motion is a second order differential equation.

Although first order differential equations are in general easier to solve than second order differential equations, the Hamiltonian is often more difficult to construct than the Lagrangian since we must express the Hamiltonian in terms of the generalized position and the generalized momentum.

Central-Force Motion

In this Chapter we will use the theory we have discussed in Chapter 6 and 7 and apply it to very important problems in physics, in which we study the motion of two-body systems on

which central force are acting. We will encounter important examples from astronomy and from nuclear physics.

Two-Body Systems with a Central Force

Consider the motion of two objects that are effected by a force acting along the line connecting the centers of the objects. To specify the state of the system, we must specify six coordinates (for example, the (x, y, z) coordinates of their centers). The Lagrangian for this system is given by

$$L = \frac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\vec{r}}_2\right|^2 - U(\vec{r}_1 - \vec{r}_2)$$

Note: here we have assumed that the potential depends on the position vector between the two objects. This is not the only way to describe the system; we can for example also specify the position of the center-of-mass, R, and the three components of the relative position vector r. In this case, we choose a coordinate system such that the center-of-mass is at rest, and located at the origin. This requires that

$$\vec{R} = \frac{m_1}{m_1 + m_2}\vec{r}_1 + \frac{m_2}{m_1 + m_2}\vec{r}_2 = 0$$

The relative position vector is defined as

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

The position vectors of the two masses can be expressed in terms of the relative position vector:

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}$$

$$\vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}$$

The Lagrangian can now be rewritten as

$$L = \frac{1}{2}m_1\left(\frac{m_2}{m_1 + m_2}\right)^2\left|\dot{\vec{r}}\right|^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1 + m_2}\right)^2\left|\dot{\vec{r}}\right|^2 - U(\vec{r}) = \frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2 - U(\vec{r})$$

where μ is the reduced mass of the system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Two-Body Systems with a Central Force: Conserved Quantities

Since we have assumed that the potential U depends only on the relative position between the two objects, the system poses spherical symmetry. As we have seen in Chapter 7, this type of symmetry implies that the angular momentum of the system is conserved. As a result, the momentum and position vector will lay in a plane, perpendicular to the angular momentum vector, which is fixed in space. The three-dimensional problem is thus reduced to a two-dimensional problem. We can express the Lagrangian in terms of the radial distance r and the polar angle θ:

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

The generalized momenta for this Lagrangian are

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

The Lagrange equations can be used to determine the derivative of these momenta with respect to time:

$$\dot{p}_r = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0$$

The last equation tells us that the generalized momentum p_θ is constant:

$$l = \mu r^2\dot{\theta} = \text{constant}$$

The constant l is related to the areal velocity. Consider the situation in Figure 1. During the time interval dt, the radius vector sweeps an area dA where

$$dA = \frac{1}{2}r^2 d\theta$$

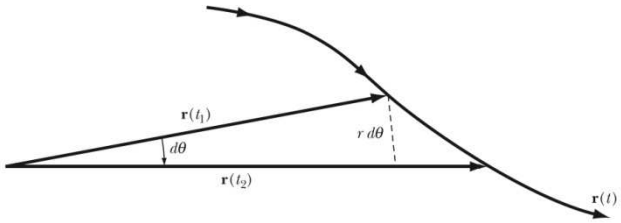


Figure 1. Calculation of the areal velocity.

The areal velocity, dA/dt, is thus equal to

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{l}{\mu r^2} = \frac{l}{2\mu} = \text{constant}$$

This result is also known as **Kepler's Second Law**.

The Lagrangian for the two-body system does not depend explicitly on time. In Chapter 7 we showed that in that case, the energy of the system is conserved. The total energy E of the system is equal to

$$E = T + U = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\left(\frac{l}{\mu r^2}\right)^2\right) + U(r) =$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

Two-Body Systems with a Central Force: Equations of Motion

If the potential energy is specified, we can use the expression for the total energy E to determine dr/dt:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}}$$

This equation can be used to find the time t as function of r:

$$t = \int dt = \pm\int \frac{1}{\sqrt{\frac{2}{\mu}(E - U(r)) - \frac{l^2}{\mu^2 r^2}}} dr$$

However, in many cases, the shape of the trajectory, $\theta(r)$, is more important than the time dependence. We can express the change in the polar angle in terms of the change in the radial distance:

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

Integrating both sides we obtain the following orbital equation

$$\theta(r) = \int \frac{\dot{\theta}}{\dot{r}}dr = \pm\int \frac{\frac{l}{\mu r^2}}{\sqrt{\frac{2}{\mu}(E-U)-\frac{l^2}{\mu^2 r^2}}}dr = \pm\int \frac{l}{r^2\sqrt{2\mu\left(E-U-\frac{l^2}{2\mu r^2}\right)}}dr$$

The extremes of the orbit can be found in general by requiring that $dr/dt = 0$, or

$$\frac{2}{\mu}(E-U)-\frac{l^2}{\mu^2 r^2} = \frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right) = 0$$

In general, this equation has two solutions, and the orbit is confined between a minimum and maximum value of r. Under certain conditions, there is only a single solution, and in that case the orbit is circular. Using the orbital equation we can determine the change in the polar angle when the radius changes from r_{min} to r_{max}. During one period, the polar angle will change by

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{l}{r^2\sqrt{2\mu\left(E-U-\frac{l^2}{2\mu r^2}\right)}}dr$$

If the change in the polar angle is a rational fraction of 2π then after a number of complete orbits, the system will have returned to its original position. In this case, the orbit is closed. In all other cases, the orbit is open.

The orbital motion is specified above in terms of the potential U. Another approach to study the equations of motion is to start from the Lagrange equations. In this case we obtain an equation of motion that includes the force F instead of the potential U:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)$$

This version of the equations of motion is useful when we can measure the orbit and want to find the force that produces this orbit.

Solving the Orbital Equation

The orbital equation can only be solved analytically for certain force laws. Consider for example the gravitational force. The corresponding potential is $-k/r$ and the polar angle θ is thus equal to

$$\theta(r) = \pm\int \frac{l/r^2}{\sqrt{2\mu\left(E+\frac{k}{r}-\frac{l^2}{2\mu r^2}\right)}}dr$$

Consider the change of variables from r to $u = l/r$:

$$\theta(r) = \pm\int \frac{(l/r)^2/l}{\sqrt{2\mu\left(E+\frac{k}{l}\frac{l}{r}-\frac{1}{2\mu}\left(\frac{l}{r}\right)^2\right)}}d\left(\frac{l}{u}\right) = \pm\int \frac{u^2/l}{\sqrt{2\mu\left(E+\frac{k}{l}u-\frac{1}{2\mu}u^2\right)}}\frac{l}{u^2}du =$$

$$= \pm\int \frac{1}{\sqrt{2\mu\left(E+\frac{k}{l}u-\frac{1}{2\mu}u^2\right)}}du$$

The integral can be solved using one of the integrals found in Appendix E (see E8.c):

$$\theta(r) = -\int \frac{1}{\sqrt{-u^2+2\mu\frac{k}{l}u+2\mu E}}du = -\sin^{-1}\left(\frac{-2u+2\mu\frac{k}{l}}{\sqrt{\left(2\mu\frac{k}{l}\right)^2-8\mu E}}\right)+C =$$

$$= \pm\sin^{-1}\left(\frac{\mu\frac{k}{l}-u}{\sqrt{\left(\mu\frac{k}{l}\right)^2+2\mu E}}\right)+C = \pm\sin^{-1}\left(\frac{\mu\frac{k}{l}-\frac{l}{r}}{\sqrt{\left(\mu\frac{k}{l}\right)^2+2\mu E}}\right)+C =$$

$$= \pm\sin^{-1}\left(\frac{\mu k-\frac{l^2}{r}}{\sqrt{(\mu k)^2+2\mu l^2 E}}\right)+C$$

This equation can be rewritten as

$$\sin(\theta + \text{constant}) = \frac{\mu k - \frac{l^2}{r}}{\sqrt{(\mu k)^2 + 2\mu l^2 E}}$$

We can always choose our reference position such that the constant is equal to $\pi/2$ and we thus find the following solution:

$$\cos(\theta) = \frac{\mu k - \frac{l^2}{r}}{\sqrt{(\mu k)^2 + 2\mu l^2 E}}$$

We can rewrite this expression such that we can determine the distance r as function of the polar angle:

$$r = \frac{l^2}{\mu k - \sqrt{(\mu k)^2 + 2\mu l^2 E}\cos(\theta)} = \frac{l^2}{\mu k\left(1 - \sqrt{1 + \frac{2}{\mu}\frac{l^2}{k^2}E}\cos(\theta)\right)}$$

Since $\cos\theta$ varies between -1 and +1, we see that the minimum (the **pericenter**) and the maximum (the **apocenter**) positions are

$$r_{min} = \frac{l^2}{\mu k\left(1 + \sqrt{1 + \frac{2}{\mu}\frac{l^2}{k^2}E}\right)}$$

$$r_{max} = \frac{l^2}{\mu k\left(1 - \sqrt{1 + \frac{2}{\mu}\frac{l^2}{k^2}E}\right)}$$

The equation for the orbit is in general expressed in terms of the **eccentricity** ε and the **latus rectum** 2α:

$$\varepsilon = \sqrt{1 + \frac{2}{\mu}\frac{l^2}{k^2}E}$$

$$\alpha = \frac{l^2}{\mu k}$$

The possible orbits are usually parameterized in terms of the eccentricity, and examples are shown Figure 2.

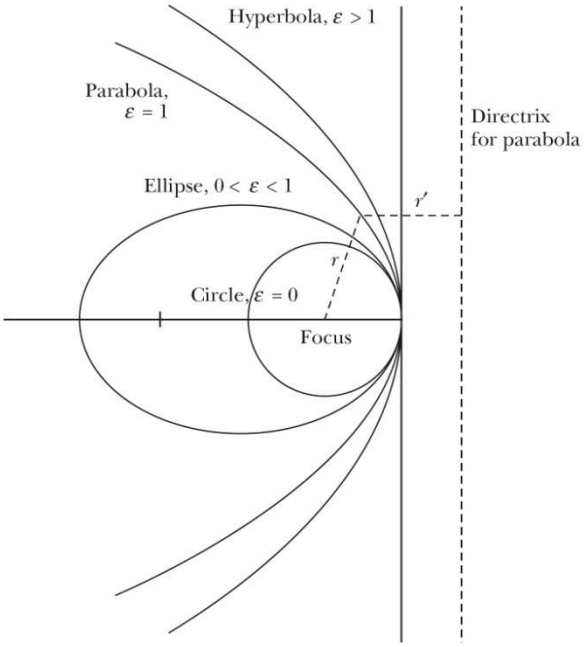


Figure 2. Possible orbits in the gravitational field.

The period of the orbital motion can be found by integrating the expression for dt over one complete period:

$$\tau = \int dt = \frac{2\mu}{l}\int dA = \frac{2\mu}{l}(\pi ab) = \frac{2\mu}{l}\left(\pi \frac{k}{2|E|}\frac{l}{\sqrt{2\mu|E|}}\right) = \pi k\sqrt{\frac{\mu}{2}}|E|^{-3/2}$$

When we take the square of this equation we get Kepler's third law:

$$\tau^2 = \pi^2 k^2 \frac{\mu}{2}|E|^{-3} = \pi^2 k^2 \frac{\mu}{2}\left(\frac{2a}{k}\right)^3 = \frac{4\pi^2\mu}{k}a^3$$

The Centripetal Force and Potential

In the previous discussion it appears as if the potential U is modified by the term $l^2/(2\mu r^2)$. This term depends only on the position r since l is constant, and it is interpreted as a potential energy. The force associated with this potential energy is

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 = \mu\frac{(r\dot{\theta})^2}{r}$$

This force is often called the **centripetal force** (although it is not a real force), and the potential is called the **centripetal potential**. This potential is a fictitious potential and it represents the effect of the angular momentum about the origin. Figure 3 shows an example of the real potential, due to the gravitational force in this case, and the centripetal potential. The effective potential is the sum of these two potentials and has a characteristic dip where the potential energy has a minimum. The result of this dip is that there are certain energies for which the orbit is bound (has a minimum and maximum distance). These turning points are called the **apsidal distances** of the orbit.

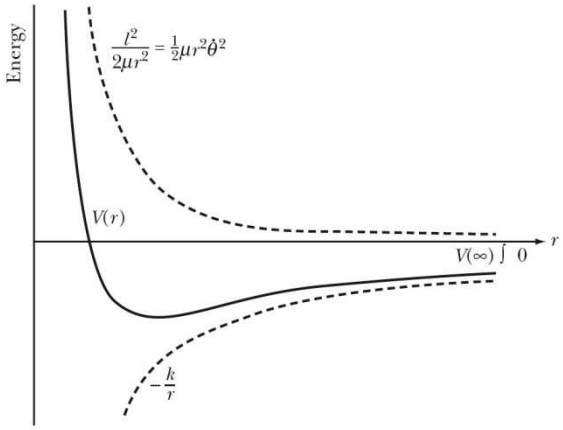


Figure 3. The effective potential for the gravitational force when the system has an angular momentum *l*.

We also note that at small distances the force becomes repulsive.

Orbital Motion

The understanding of orbital dynamics is very important for space travel. The orbit in which a spaceship travels is determined by the energy of the spaceship. When we change the energy of the ship, we will change the orbit from for example a spherical orbit to an elliptical orbit. By changing the velocity at the appropriate point, we can control the orientation of the new orbit.

The Hofman transfer represents the path of minimum energy expenditure to move from one solar-based orbit to another. Consider travel from earth to mars (see Figure 4). The goal is to get our spaceship in an orbit that has apsidal distances that correspond to the distance between the earth and the sun and between mars and the sun. This requires that

$$r_1 = a(1-\varepsilon)$$

and

$$r_2 = a(1+\varepsilon)$$

The eccentricity of such an orbit is thus equal to

$$\varepsilon = \frac{r_2 - r_1}{2a}$$

The total energy of an orbit with a major axis of $a = (r_1 + r_2)/2$ is equal to

$$E = \frac{k}{2a} = \frac{k}{(r_1 + r_2)}$$

Since the space ship starts from a circular orbit with a major axis $a = r_1$, its initial energy is equal to

$$E = -\frac{k}{2r_1}$$

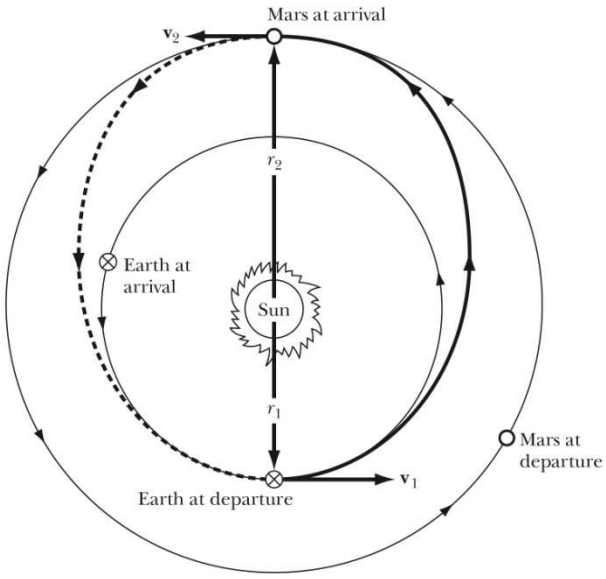


Figure 4. The Hofman transfer to travel from earth to mars.

The increase in the total energy is thus equal to

$$\Delta E = -\frac{k}{(r_1 + r_2)} - \left(-\frac{k}{2r_1}\right) = \left(\frac{k}{2}\right)\left\{\frac{1}{r_1} - \frac{2}{(r_1 + r_2)}\right\} = \left(\frac{k}{2}\right)\frac{r_2 - r_1}{r_1(r_1 + r_2)}$$

This energy must be provided by the thrust of the engines that increase the velocity of the space ship (note: the potential energy does not change at the moment of burn, assuming the thrusters are only fired for a short period of time).

The problem with the Hofman transfer mechanism is that the conditions have to be just right, and only of the planets are in the proper position will the transfer work. There are many other ways to travel between earth and mars. Many of these require less time than the time required for the Hofman transfer, but they require more fuel (see Figure 5).

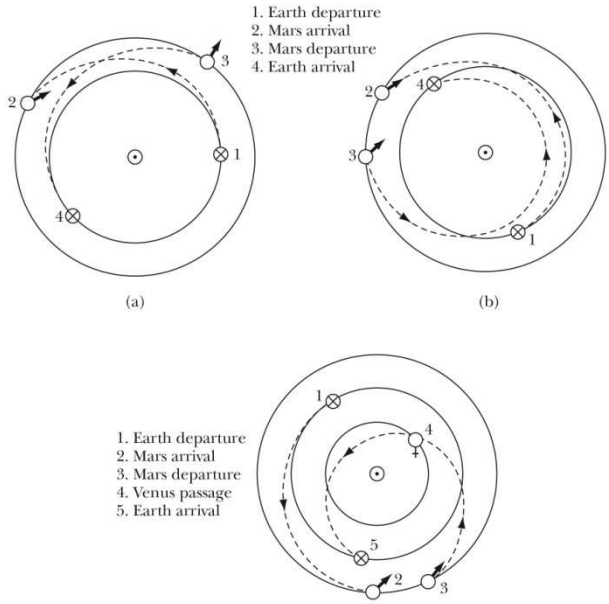


Figure 5. Different ways to get from earth to mars.

Dynamics of a System of Particles

In this Chapter we expand our discussion from the two-body systems discussed in Chapter 8 to systems that consist out of many particles. In general, these particles are exposed to both external and internal forces. In our discussion in the Chapter we will make the following assumptions about the internal forces:

- • The forces exerted between any two particles are equal in magnitude and opposite in direction.
- • The forces exerted between any two particles are directed parallel or anti-parallel to the line joining the two particles.

These two requirements are fulfilled for many forces. However, there are important forces, such as the magnetic force, do not satisfy the second assumption.

The Center-of-Mass

As we discussed in Chapter 9, it is often useful to separate the motion of a system into the motion of its center of mass and the motion of its component relative to the center of mass. The definition of the position of the center of mass for a multi-particle system (see Figure 1) is similar to its definition for a two-body system:

$$\bar{R}_{cm} = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \bar{r}_i$$

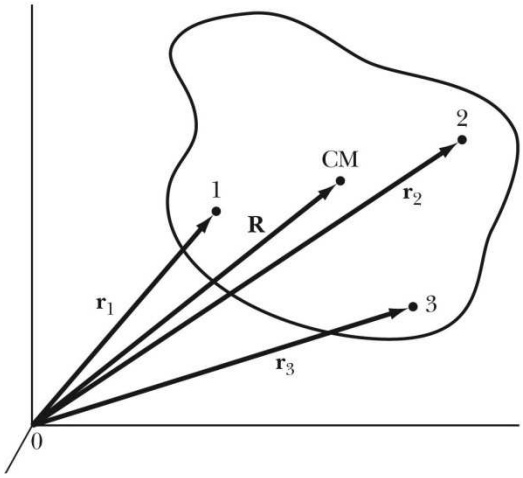


Figure 1. The location of the center of mass of a multi-particle system.

If the mass distribution is a continuous distribution, the summation is replaced by an integration:

$$\bar{R}_{cm} = \frac{1}{M}\int \bar{r}\,dm$$

Linear Momentum

Consider a system of particles, of total mass *M*, exposed to internal and external forces. The linear momentum for this system is defined as

$$\vec{P} = \sum_i m_i \vec{v}_i = \frac{d}{dt}\sum_i m_i \vec{r}_i = \frac{d}{dt}\left(M\vec{R}\right) = M\vec{V}$$

The change in the linear momentum of the system can be expressed in terms of the forces acting on all the particles that make up the system:

$$\frac{d\vec{P}}{dt} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i = \sum_i \vec{F}_{i,ext} + \sum_i \sum_{j \neq i} \vec{F}_{ij,int} = \vec{F}_{ext}$$

We see that the linear momentum is constant if the net external force acting on the system is 0 N. If there is an external force acting on the system, the component of the linear momentum in the direction of the net external force is not conserved, but the components in the directions perpendicular to the direction of the net external force are conserved.
We conclude that the linear momentum of the system has the following properties:

1. 1. The center of mass of a system moves as if it were a single particle with a mass equal to the total mass of the system, *M*, acted on by the total external force, and independent of the nature of the internal forces.

2. 2. The linear momentum of a system of particles is the same as that of a single particle of mass *M*, located at the position of the center of mass, and moving in the manner the center of mass is moving.

3. 3. The total linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass.

Angular Momentum

Consider a system of particles that are distributed as shown in Figure 2. We can specify the location of the center of mass of this system by specifying the vector *R*. This position vector may be time dependent. The location of each component of this system can be specified by either specifying the position vector, r_α, with respect to the origin of the coordinate system, or by specifying the position of the component with respect to the center of mass, r_α'.

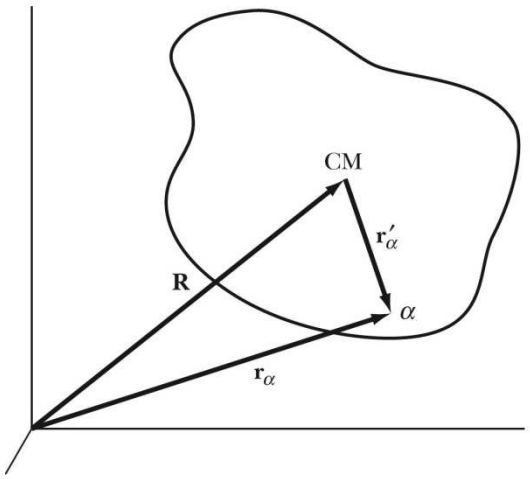


Figure 2. Coordinate system used to describe a system of particles.

The angular momentum of this system with respect to the origin of the coordinate system is equal to

$$\bar{L} = \sum_\alpha \bar{L}_\alpha = \sum_\alpha \left\{ \bar{r}_\alpha \times m_\alpha \dot{\bar{r}}_\alpha \right\} = \sum_\alpha \left\{ \left(\bar{R} + \bar{r}'_\alpha \right) \times m_\alpha \left(\dot{\bar{R}} + \dot{\bar{r}}'_\alpha \right) \right\}$$

Since

$$\sum_\alpha \left\{ m_\alpha \dot{\bar{r}}'_\alpha \right\} = \sum_\alpha \left\{ m_\alpha \left(\dot{\bar{r}}_\alpha - \dot{\bar{R}} \right) \right\} = \sum_\alpha \left\{ m_\alpha \dot{\bar{r}}_\alpha \right\} - \dot{\bar{R}} \sum_\alpha m_\alpha = M\dot{\bar{R}} - M\dot{\bar{R}} = 0$$

we can rewrite the expression for the angular momentum as

$$\bar{L} = \left(\bar{R} \times \dot{\bar{R}} \right) \sum_\alpha m_\alpha + \sum_\alpha \left\{ \bar{r}'_\alpha \times m_\alpha \dot{\bar{r}}'_\alpha \right\} = \bar{R} \times \bar{P} + \sum_\alpha \left\{ \bar{r}'_\alpha \times \bar{p}'_\alpha \right\} = \bar{L}_{cm} + \bar{L}_{w.r.cm}$$

The angular momentum is thus equal to sum of the angular momentum of the center of mass and the angular momentum of the system with respect to the center of mass.

The rate of change in the angular momentum of the system can be determined by using the following relation:

$$\frac{d\bar{L}}{dt} = \dot{\bar{L}} = \frac{d}{dt}\left(\bar{r} \times \bar{p} \right) = \dot{\bar{r}} \times \bar{p} + \bar{r} \times \dot{\bar{p}} = \bar{r} \times \dot{\bar{p}} = \bar{r} \times \bar{F}$$

For the system we are currently discussing we can thus conclude that

$$\frac{d\bar{L}}{dt}=\sum_i\left\{\bar{r}_i\times\bar{F}_{i,ext}\right\}+\sum_i\sum_{j\neq i}\left\{\bar{r}_i\times\bar{F}_{ij,int}\right\}=\sum_i\left\{\bar{r}_i\times\bar{F}_{i,ext}\right\}+\sum_i\sum_{j<i}\left\{\left(\bar{r}_i-\bar{r}_j\right)\times\bar{F}_{ij,int}\right\}=\sum_i\left\{\bar{r}_i\times\bar{F}_{i,ext}\right\}$$

The last step in this derivation is only correct of the internal force between *i* and *j* is parallel or anti-parallel to the relative position vector, but this was one of the two assumptions we made about the internal forces at the beginning of this Chapter. Since the vector product between the position vector and the force vector is the torque *N* associated with this force, we can rewrite the rate of change of the angular momentum of the system as

$$\frac{d\bar{L}}{dt}=\sum_i\bar{N}_{i,ext}$$

We conclude that the angular momentum of the system has the following properties:

1. 1. The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.

2. 2. If the net resultant torques about a given axis vanish, then the total angular momentum of the system about that axis remained constant in time.

3. 3. The total internal torque must vanish if the internal forces are central, and the angular momentum of an isolated system can not be altered without the application of external forces.

Energy

The total energy of a system of particles is equal to the sum of the kinetic and the potential energy.

The kinetic energy of the system is equal to the sum of the kinetic energy of each of the components. The kinetic energy of particle *i* can either be expressed in terms of its velocity with respect to the origin of the coordinate system, or in terms of its velocity with respect to the center of mass:

$$r_i^2=\bar{r}_i\bullet\bar{r}_i=\left(\bar{R}+\bar{r}_i'\right)\bullet\left(\bar{R}+\bar{r}_i'\right)=\bar{R}\bullet\bar{R}+2\bar{r}_i'\bullet\bar{R}+\bar{r}_i'\bullet\bar{r}_i'$$

The kinetic energy of the system is thus equal to

$$T=\sum_i\frac{1}{2}m_iv_i^2=\sum_i\frac{1}{2}m_iv_i'^2+\sum_i m_i\left(\bar{v}_i'\bullet\bar{V}\right)+\sum_i\frac{1}{2}m_iV^2=\sum_i\frac{1}{2}m_iv_i'^2+\bar{V}\bullet\frac{d}{dt}\sum_i m_i\bar{r}_i'+\frac{1}{2}MV^2$$

Based on the definition of the position of the center of mass:

$$\bar{R} = \frac{1}{M}\sum_i m_i \bar{r}_i = \frac{1}{M}\sum_i m_i \left(\bar{R} + \bar{r}_i'\right) = \bar{R} + \frac{1}{M}\sum_i m_i \bar{r}_i'$$

we conclude that

$$\sum_i m_i \bar{r}_i' = 0$$

The kinetic energy of the system is thus equal to

$$T = \frac{1}{2}MV^2 + \sum_i \frac{1}{2} m_i v_i'^2$$

The change in the potential energy of the system when it moves from a configuration 1 to a configuration 2 is related to the work done by the forces acting on the system:

$$W_{12} = \sum_i \int_1^2 \bar{F}_{i,ext} \cdot d\bar{r}_i + \sum_i \int_1^2 \sum_{j \neq i} \bar{F}_{ij,int} \cdot d\bar{r}_i$$

If we make the assumption that the forces, both internal and external, are derivable from potential functions, we can rewrite this expression as

$$W_{12} = -\sum_i \int_1^2 \bar{\nabla}\left(U_{i,ext}\right) \cdot d\bar{r}_i - \sum_i \int_1^2 \sum_{j \neq i} \bar{\nabla}\left(U_{ij,int}\right) \cdot d\bar{r}_i$$

The first term on the right-hand side can be evaluated easily:

$$\sum_i \int_1^2 \bar{\nabla}\left(U_{i,ext}\right) \cdot d\bar{r}_i = \sum_i U_{i,ext}\Big|_1^2 = \sum_i \left\{U_{i,ext}(2) - U_{i,ext}(1)\right\}$$

The second term can be rewritten as

$$\sum_i \int_1^2 \sum_{j \neq i} \bar{\nabla}\left(U_{ij,int}\right) \cdot d\bar{r}_i = \sum_i \int_1^2 \sum_{j<i} \left\{\bar{\nabla}\left(U_{ij,int}\right) \cdot d\bar{r}_i + \bar{\nabla}\left(U_{ji,int}\right) \cdot d\bar{r}_j\right\} = \sum_i \int_1^2 \sum_{j<i} \left\{\bar{\nabla}\left(U_{ij,int}\right) \cdot d\bar{r}_{ij}\right\}$$

Here we have used the fact that the internal force between *i* and *j* satisfy the following relation

$$\vec{F}_{ij,int} = -\vec{F}_{ji,int}$$

and thus

$$U_{ij,int} = -U_{ji,int}$$

The integral can be evaluated easily:

$$\sum_i \int_1^2 \sum_{j \neq i} \left\{ \vec{\nabla}\left(U_{ij,int}\right) \cdot d\vec{r}_{ij} \right\} = \sum_i \sum_{j \neq i} \int_1^2 dU_{ij,int} = \sum_i \sum_{j \neq i} \left\{ U_{ij,int}(2) - U_{ij,int}(1) \right\}$$

The total potential energy of the system U is defined as the sum of the internal and the external potential energy and is equal to

$$U = \sum_i U_{i,ext} + \sum_i \sum_{j \neq i} U_{ij,int}$$

The work done by al the force to make the transition from configuration 1 to configuration 2 is

$$W_{12} = -U\Big|_1^2 = -\left(U_2 - U_1\right) = U_1 - U_2$$

Using the work-energy theorem we can conclude that

$$W_{12} = U_1 - U_2 = \Delta T = T_2 - T_1$$

or

$$E_1 = T_1 + U_1 = T_2 + U_2 = E_2$$

We thus see that the total energy is conserved. If the system of particles is a rigid object, the components of the system will retain their relative positions, and the internal potential energy of the system will remain constant.
We conclude that the total energy of the system has the following properties:

1. 1. The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with the velocity of the center of mass and the kinetic energy of the motion of the individual particles relative to the center of mass.

2. 2. The total energy for a conservative system is constant.

Elastic and Inelastic Collisions

When two particles interact, the outcome of the interaction will be governed by the force law that describes the interaction. Consider an interaction force F_{int} that acts on a particle. The result of the interaction will be a change in the momentum of the particle since

$$F_{int} = ma = m\frac{dv}{dt} = \frac{dp}{dt}$$

If the interaction occurs over a short period of time, we expect to a change in the linear momentum of the particle:

$$\int_i^f \vec{F}_{int}dt = \int_i^f \left(\frac{d\vec{p}}{dt}\right)dt = \vec{p}_f - \vec{p}_i = \Delta\vec{p}$$

This relation shows us that if we know the force we can predict the change in the linear momentum, or if we measure the change in the linear momentum we can extract information about the force. We note that the change in the linear momentum provides us with information about the time integral of the force, not the force. Due to the importance of the time integral, it has received its own name, and is called the impulse P:

$$\vec{P} = \int_i^f \vec{F}_{int}dt$$

If we consider the effect of the interaction force on both particles we conclude that the change in the linear momentum is 0:

$$\Delta\vec{p} = \Delta\vec{p}_1 + \Delta\vec{p}_2 = \int_i^f \vec{F}_{(2,int)}dt + \int_i^f \vec{F}_{21,int}dt = \int_i^f \vec{F}_{(2,int)}dt - \int_i^f \vec{F}_{12,int}dt = 0$$

This of course should be no surprise since when we consider both particles, the interaction force becomes an internal force and in the absence of external forces, linear momentum will be conserved.

The conservation of linear momentum is an important conservation law that restricts the possible outcomes of a collision. No matter what the nature of the collision is, if the initial linear momentum is non-zero, the final linear momentum will also be non-zero, and the system can not be brought to rest as a result of the collision. If the system is at rest after the collision, its linear momentum is zero, and the initial linear momentum must therefore also be equal to zero. Note that a zero linear momentum does not imply that all components of the system will be at rest; it only requires that the two object have linear momenta that are equal in magnitude but directed in opposite directions.

The most convenient way to look at the collisions is in the center-of-mass frame. In the center-of-mass frame, the total linear momentum is equal to zero, and the objects will always travel in a co-linear fashion. This illustrated in Figure 4.

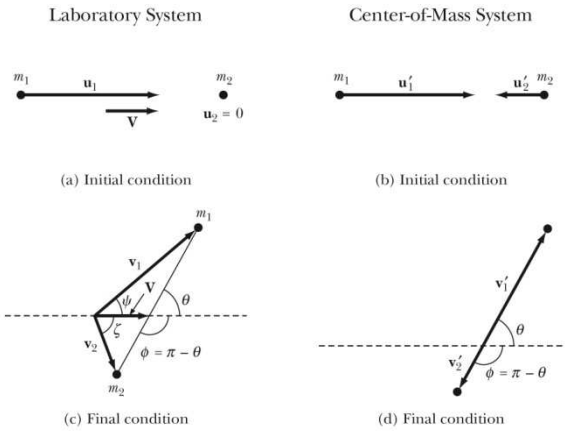


Figure 4. Two-dimensional collisions in the laboratory frame (left) and the center-of-mass frame (right).

We frequently divide collisions into two distinct groups:

- • Elastic collisions: collisions in which the total kinetic energy of the system is conserved. The kinetic energy of the objects will change as a result of the interaction, but the total kinetic energy will remain constant. The kinetic energy of one of the objects in general is a function of the masses of the two objects and the scattering angle.

- • Inelastic collisions: collisions in which the total kinetic energy of the system is not conserved. A totally inelastic collision is a collision in which the two objects after the collision stick together. The loss in kinetic energy is usually expressed in terms of the Q value, where $Q = K_f - K_i$:
    - ○ $Q < 0$: **endoergic collision** (kinetic energy is lost)
    - ○ $Q > 0$: **exoergic collision** (kinetic energy is gained)
    - ○ $Q = 0$: **elastic collision** (kinetic energy is conserved).

  In most inelastic collisions, a fraction of the initial kinetic energy is transformed into internal energy (for example in the form of heat, deformation, etc.).
  Another parameter that is frequently used to quantify the inelasticity of an inelastic collision is the coefficient of restitution ε:

$$\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$$

where u are the velocities after the collision and v are the velocities before the collision. For a perfect elastic collision $\varepsilon = 1$ and for a totally inelastic collision $\varepsilon = 0$.

One important issue we need to address when we focus on collisions is the issue of predictability. Let's consider what we know and what we need to know; we will assume that we looking at a collision in the center-of-mass frame. Let's define the x axis to be the axis parallel to the direction of motion of the incident objects, and let's assume that the masses of the objects do not change. The unknown parameters are the velocities of the object; for the n-dimensional case, there will be 2n unknown. What do we know?

- • Conservation of linear momentum: this conservation law provides us with n equations with 2n unknown.

- • Conservation of energy: if the collision is elastic, this conservation law will provide us with 1 equation with 2n unknown.

For elastic collisions we thus have n+1 equations with 2n unknown. We immediately see that only for n = 1 the final state is uniquely defined. For inelastic collisions we have n equations with 2n unknown and we conclude that event for n = 1 the final state is undefined. When the final state is undefined we need to know something about some of the final-state parameters to fix the others.

There are many applications of our collision theory. Consider one technique that can be used to study the composition of a target material. We use a beam of particles of known mass m_1 and kinetic energy T_{initial} to bombard the target material and measure the energy of the elastically scattered projectiles at 90°. The measured kinetic energy depends on the mass of the target nucleus and is given by

$$T_{final} = \frac{m_1^2}{m_1^2 + m_2^2}\left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]T_{initial}$$

By measuring the final kinetic energy we can thus determine the target mass. Note: we need to make sure that the object we detect at 90° is the projectile. An example of an application of this is shown in Figure 5.

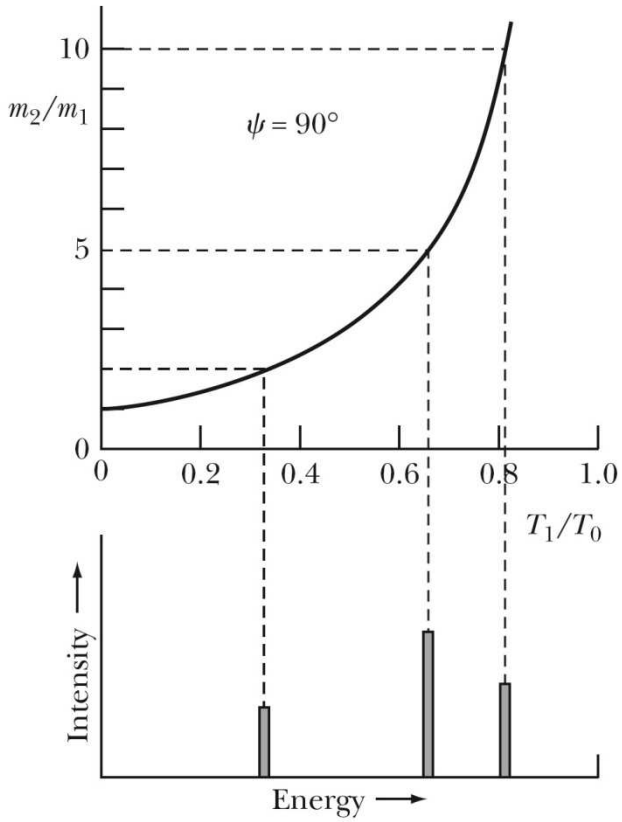


Figure 5. Energy spectrum of the scattered projectiles at 90°.

Scattering Cross Section

We have learned a lot of properties of atoms and nuclei using elastic scattering of projectiles to probe the properties of the target elements. A schematic of the scattering process is shown in Figure 6. In this Figure, the incident particle is deflected (repelled) by the target particle. This situation will arise when we consider the scattering of positively charged nuclei.

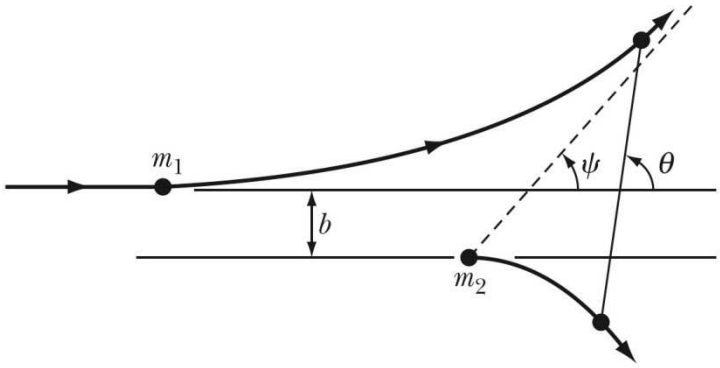


Figure 6. Scattering of projectile nuclei from target nuclei.

The parameter b is called the **impact parameter**. The impact parameter is related to the angular momentum of the projectile with respect to the target nucleus.

When we study the scattering process we in general measure the intensity of the scattered particles as function of the scattering angle. The intensity distribution is expressed in terms of the differential cross section, which is defined as

$$\sigma(\theta) = \frac{\#\text{of interactions per target nucleus into an area } d\Omega' \text{ at } \theta}{\#\text{ of incident particles per units area}}$$

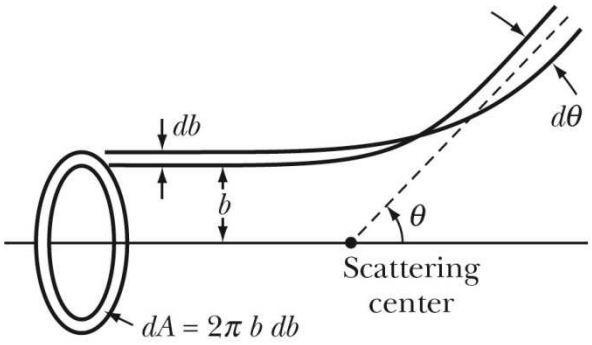


Figure 7. Correlation between impact parameter and scattering angle.

There is a one-to-one correlation between the impact parameter b and the scattering angle θ (see Figure 7). The one-to-one correlation is a direct effect of the conservation of angular momentum. Assuming that the number of incident particles is conserved, the flux of incident particles with an impact parameter between b and $b + db$ is equal to

$$dI_{in} = I\{2\pi b db\}$$

must be the same as the number of particles scattered in the cone that is specified by the angle θ and width $d\theta$. The area of this cone is equal to

$$d\Omega' = 2\pi \sin\theta d\theta$$

The number of particles scattered into this cone will be

$$dI_{out} = -I\sigma(\theta)2\pi \sin\theta d\theta$$

The minus sign is a result of the fact that if $db > 0$, $d\theta < 0$. We thus conclude that

$$I\{2\pi b db\} = -I\sigma(\theta)2\pi \sin\theta d\theta$$

or

$$\sigma(\theta)=\frac{\{2\pi b db\}}{\{-2\pi\sin\theta d\theta\}}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

The scattering angle θ is related to the impact parameter b and this relation can be obtained using the orbital motion we have discussed in this and in the previous Chapter:

$$\theta=\int_{r_{min}}^{\infty}\frac{(b/r^2)dr}{\sqrt{1-(b^2/r^2)-(U/T_0')}}$$

The relation between the scattering angle θ and the impact parameter b depends on the potential U. For the important case of nuclear scattering, the potential varies as k/r. For this potential we can carry out the integration and determine the following correlation between the scattering angle and the impact parameter:

$$\cos\theta=\frac{(\kappa/b)}{\sqrt{1+(\kappa/b)^2}}$$

where

$$\kappa=\frac{k}{2T_0'}$$

We can use this relation to calculate $db/d\theta$ and get the following differential cross section:

$$\sigma(\theta)=\left(\frac{k}{2T_0'}\right)^2\frac{1}{\sin^4(\theta/2)}$$

We conclude that the intensity of scattered projectile nuclei will decrease when the scattering angle increases. If the energy of projectile nuclei is low enough, the measured angular distribution will agree with the so-called Rutherford distribution over the entire angular range, as was first shown by Geiger and Marsden in 1913 (see Figure 8 left).

Each trajectory of the projectile can be characterized by distance of closest approach and there is a one-to-one correspondence between the scattering angle and this distance of closest approach. The smallest distance of closest approach occurs when the projectile is scattered backwards (θ = 180°). The distance of closest approach decreases with increasing incident energy and the Rutherford formula indicates that the intensity should decrease as $1/T^2$. This was indeed observed, up to a maximum incident energy, beyond which the intensity dropped of much more rapidly than predicted by the Rutherford formula (see Figure 8 right). At this point, the nuclei approach each other so close that the strong attractive nuclear force starts to play a role,

and the scattering is no longer elastic (the projectile nuclei may for example merge with the target nuclei).

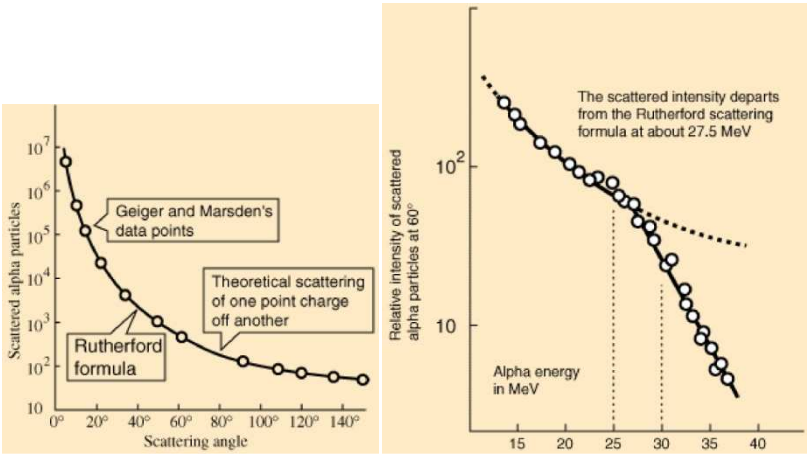

Scattered alpha particles
Geiger and Marsden's data points
Theoretical scattering of one point charge off another
Rutherford formula
Scattering angle
Relative intensity of scattered alpha particles at 60°
The scattered intensity departs from the Rutherford scattering formula at about 27.5 MeV
Alpha energy in MeV

Figure 8. Measurement of the scattering of alpha particles from target nuclei. Figures taken from http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html.

Motion in a Non-Inertial Reference Frame

The laws of physics are only valid in inertial reference frames. However, it is not always easy to express the motion of interest in an inertial reference frame. Consider for example the motion of a book laying on top of a table. In a reference frame that is fixed with respect to the Earth, the motion is simple: if the book is at rest, it will remain at rest (here we assume that the surface of the table is horizontal). However, we do know that the earth frame is not an inertial frame. In order to describe the motion of the book in an inertial frame, we need to take into account the rotation of the Earth around its axis, the rotation of the Earth around the sun, the rotation of our solar system around the center of our galaxy, etc. etc. The motion of the book will all of a sudden be a lot more complicated!

For many experiments, the effect of the Earth not being an inertial reference frame is too small to be observed. Other effects, such as the tides, can only be explained if we take into consideration the non-inertial nature of the reference frame of the Earth and apply the laws of physics in n inertial frame.

Rotating Coordinate Systems

Consider the two coordinate systems shown in Figure 1. The non-primed coordinates are the coordinates in the rotating frame, and the primed coordinates are the coordinates in the fixed coordinate system.

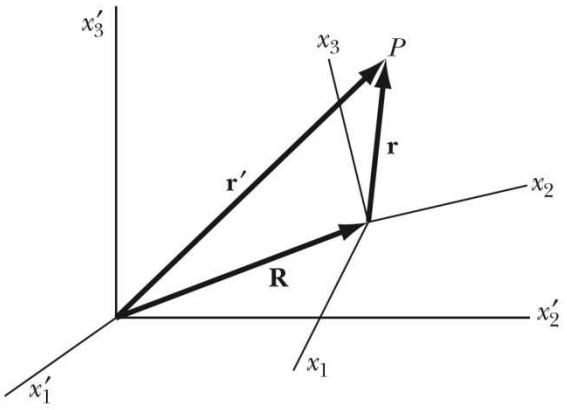


Figure 1. Fixed (primed) and rotating (non-primed) coordinate systems. The vector R specifies the origin of the rotating coordinate system in the fixed coordinate system.

Consider the motion of a point P. In the fixed coordinate system, the position of P is specified by the position vector r' and in the rotating coordinate system, its position is specified by the position vector r. As can be seen from Figure 1, these two vectors are related:

$$\bar{r}\,' = \bar{r} + \bar{R}$$

Consider what happens when the rotating coordinate system rotates by an infinitesimal angle $d\theta$. If point P is at rest in the rotating coordinate system, we will see the position of P in our fixed coordinate system change:

$$(d\bar{r})_{fixed} = d\bar{\Theta} \times \bar{r}$$

It the rotation occurs during a period dt, we can rewrite the previous equation as

$$\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \frac{d\bar{\theta}}{dt} \times \bar{r} = \bar{\omega} \times \bar{r}$$

To derive this relation we have assume that point P remains are test in the rotating coordinate system. If point P is moving with respect to the rotating coordinate system, we need to add this contribution to the expression of the velocity of P in the fixed coordinate system:

$$\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{r}$$

This relation is valid for any vector, not just the position vector. If instead of the position vector we use the angular velocity vector we find that

$$\left(\frac{d\bar{\omega}}{dt}\right)_{fixed} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{\omega} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating}$$

This relation shows that the angular acceleration is the same in both reference frames.

In order to determine the velocity of point P in the fixed coordinate frame in terms of the velocity of point P in the rotating coordinate system, we have to go back to the correlation between the position vectors shown in Figure 1. By differentiating the vectors with respect to time we obtain the following relation:

$$\left(\frac{d\bar{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{fixed}$$

Using our expression for the velocity of P in the fixed coordinate system we find that

$$\left(\frac{d\bar{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{r}$$

This equation can also be rewritten as

$$v_f = \left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r} = V + v_r + \bar{\omega}\times\bar{r}$$

where

$$\bar{v}_f = \left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \text{velocity of P in fixed frame}$$

$$\bar{V} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} = \text{linear velocity of the origin of the rotating frame}$$

$$\bar{v}_r = \left(\frac{d\bar{r}}{dt}\right)_{rotating} = \text{velocity of P in rotating frame}$$

$$\bar{\omega}\times\bar{r} = \text{velocity of P due to the rotation of the axes}$$

"Newton's Law" in Rotating Reference Frames

Consider the situation in which an external force F is acting on P. Only on the fixed reference frame can we used Newton's second law to determine the corresponding acceleration of P:

$$\bar{a}_f = \left(\frac{d\bar{v}_f}{dt}\right)_{fixed} = \frac{\bar{F}}{m}$$

Another expression for the acceleration of P can be obtained by differentiating the velocity-relation obtained in the previous section with respect to time:

$$\bar{a}_f = \left(\frac{d\bar{v}_f}{dt}\right)_{fixed} = \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left(\frac{d\bar{v}_r}{dt}\right)_{fixed} + \left(\frac{d\bar{\omega}}{dt}\right)_{fixed}\times\bar{r} + \bar{\omega}\times\left(\frac{d\bar{r}}{dt}\right)_{fixed} =$$

$$= \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left\{\left(\frac{d\bar{v}_r}{dt}\right)_{rotating} + \bar{\omega}\times\bar{v}_r\right\} + \left(\frac{d\bar{\omega}}{dt}\right)_{fixed}\times\bar{r} + \bar{\omega}\times\left\{\left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r}\right\} =$$

$$= \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left(\frac{d\bar{v}_r}{dt}\right)_{rotating} + 2\bar{\omega}\times\bar{v}_r + \dot{\bar{\omega}}\times\bar{r} + \bar{\omega}\times\{\bar{\omega}\times\bar{r}\}$$

An observer in the rotating reference frame will observe an acceleration

$$\bar{a}_r = \left(\frac{d\bar{V}}{dt}\right)_{rotating}$$

This acceleration is certainly not equal to F/m, but the previous relations can be used to express the acceleration in the rotating reference frame in terms of the acceleration in the fixed reference frame:

$$\bar{a}_r = \bar{a}_f - \left(\frac{d\bar{V}}{dt}\right)_{fixed} - 2\bar{\omega}\times\bar{v}_r - \dot{\bar{\omega}}\times\bar{r} - \bar{\omega}\times\{\bar{\omega}\times\bar{r}\}$$

This relation immediately shows what has been repeated already many times: the acceleration of an object at P will be the same in two reference frames, only if one frame does not rotate with respect to the other frame (that is $\omega = 0$ rad/s and $d\omega/dt = 0$ rad/s^2) and if the frames do not accelerate with respect to the other frame.

In order to explore the implication of the relation between the acceleration of P in the rotating and in the fixed coordinate frames, we assume for the moment that the origin of the rotating reference frame is not accelerating with respect to the origin of the fixed reference frame ($dV/dt = 0$), and that the axis of the rotating reference frame are rotating with a constant angular velocity ($d\omega/dt = 0$ rad/s^2). Under these assumptions, we find that the acceleration of P in the rotating reference frame is equal to

$$\bar{a}_r = \bar{a}_f - 2\bar{\omega}\times\bar{v}_r - \bar{\omega}\times\{\bar{\omega}\times\bar{r}\}$$

The second and the third terms on the right-hand side are non-inertial terms that are introduced to correct the real force F in order to be able to use Newton-like laws in the rotating frame:

$$\bar{F}_{eff} = m\bar{a}_f - 2m\bar{\omega}\times\bar{v}_r - m\bar{\omega}\times\{\bar{\omega}\times\bar{r}\}$$

Using this effective force, an observer in the rotating frame will be able to determine the acceleration in the rotating frame by dividing this effective force by the mass of the object.

The second term on the right-hand side of the expression of the effective force is called the **Coriolis force**, and the last term on the right-hand side is called the **centripetal force**. Both of these forces are however not real forces; they are introduced in order to be able to use an equation similar to Newton's second law in non-inertial reference frames. When we try to describe an object on the surface of the earth, we need to take the effects of these artificial forces into consideration. In the next two sections we will focus on these two forces in some detail.

The Centripetal Force

The surface of the earth is a non-inertial reference frame. The biggest deviation from good "inertial" behavior is due to the rotation of the earth around its axis. In the current discussion we will thus ignore the motion of the earth around the sun, the motion of the solar system in our in our galaxy, etc. etc.

Consider a pendulum at rest in our rotating reference frame, which is at rest with respect to the surface of the earth. Since the pendulum is at rest in this rotating reference frame, its velocity v_r in this frame is zero. The effective force seen by the pendulum is thus equal to

$$\bar{F}_{eff} = m\bar{a}_f - m\bar{\omega}\times\{\bar{\omega}\times\bar{r}\} = m\{\bar{g}_0 - \bar{\omega}\times\{\bar{\omega}\times\bar{r}\}\}$$

The direction of g_0 in this equation is directly towards the center of the earth, while the direction of the non-inertial correction term is radially outwards (see Figure 2). If the angle between the position vector r and the rotation axis is equal to q, we find the magnitude of the correction term is equal to

$$|\bar{\omega}\times\{\bar{\omega}\times\bar{r}\}| = \omega^2 r\sin\theta$$

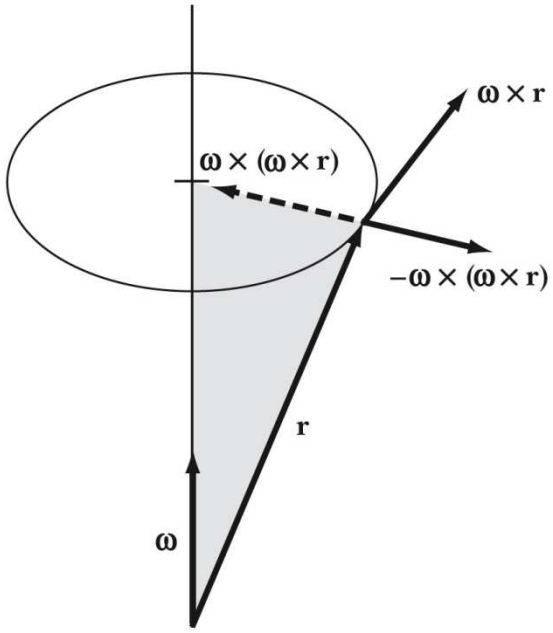


Figure 2. Direction of the centripetal correction term.

The effect of this correction is that the equilibrium position of the pendulum (the position in which the arm of the pendulum is parallel to the direction of the net force) is changed, and the arm of the pendulum no longer points towards the center of the earth (see Figure 3). The direction of the gravitational acceleration, as measured by an observer in the rotating reference frame, is thus equal to

$$\bar{g} = \bar{g}_0 - \bar{\omega} \times \{\bar{\omega} \times \bar{r}\}$$

The centripetal correction changes both the magnitude of the observed acceleration and its direction. The angle between the direction of g_0 and the direction of g can be found easily (see Figure 4):

$$\Delta\theta = \theta - \alpha = \theta - \text{atan}\left(\frac{g_0 \sin\theta - \omega^2 R \sin\theta}{g_0 \cos\theta}\right) = \theta - \text{atan}\left(\left(1 - \frac{\omega^2}{g_0}\right)\tan\theta\right)$$

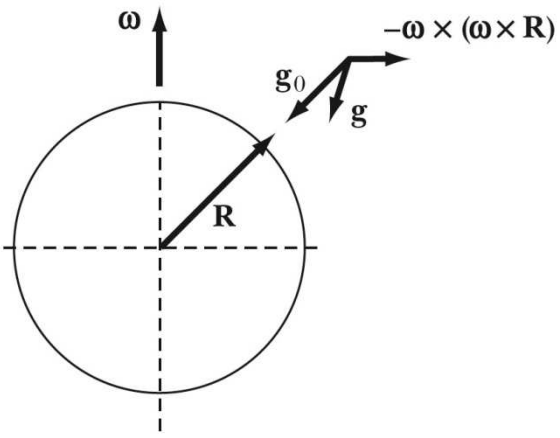


Figure 3. Effect of the centripetal term on a pendulum located on the surface of the earth.

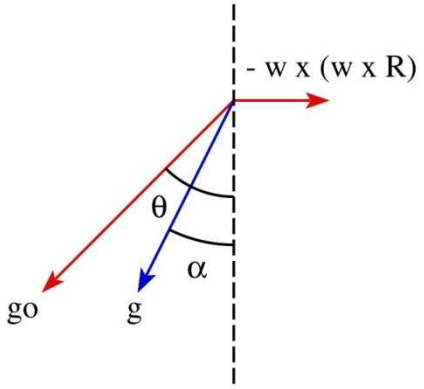


Figure 4. Direction of net gravitational acceleration.

The same result could have been obtained if we had solved this problem in a non-rotating frame. Consider a simple pendulum of mass m attached a string. There are two forces acting on this mass: the tension T in the string and the gravitational force F_g. An observer in the inertial frame will observe that mass m carries out circular motion, with a radius $R \sin\theta$, and know thus

that there must be a net force acting on it, pointing towards the rotation axis. This force must have a magnitude of

$$F_r = m\frac{v^2}{R\sin\theta} = m\frac{\left(\frac{2\pi R\sin\theta}{T}\right)^2}{R\sin\theta} = m\omega^2 R\sin\theta$$

This force must be generated by the component of the tension and the gravitational force in this direction. We must thus require that (see Figure 5):

$$mg_0\sin\theta - T\sin\alpha = m\omega^2 R\sin\theta$$

or

$$T\sin\alpha = mg_0\sin\theta - m\omega^2 R\sin\theta$$

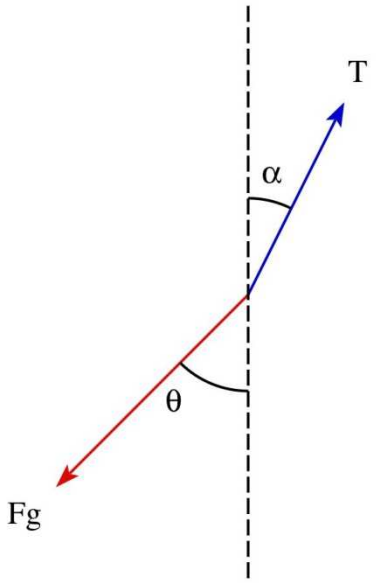


Figure 5. Pendulum in inertial frame.

The net force in the direction perpendicular to the plan of rotation must be zero, and we must thus require that

$$T\cos\alpha = mg\cos\theta$$

Combing these two equations we obtain the following relation between the angles:

$$\tan\alpha = \frac{T\sin\alpha}{T\cos\alpha} = \frac{mg_0\sin\theta - m\omega^2 R\sin\theta}{mg_0\cos\theta} = \tan\left(\left(1-\frac{\omega^2}{g_0}\right)\tan\theta\right)$$

which is the same results we obtained previously.

Coriolis Force

The Coriolis force is responsible for the deflection of objects moving in a rotating coordinate system. The force is proportional to the vector product of the angular velocity vector of the rotating coordinate system (as measured by an observer in a fixed coordinate system) and the velocity vector of the object in the rotating coordinate frame:

$$\vec{F}_{Coriolis} = -2m\left(\vec{\omega}\times\vec{v}_r\right)$$

The effect of the Coriolis force on the motion of an object is illustrated in Figure 6. Note that the deflection depends on the z component of the angular velocity vector, which is perpendicular to the surface of the earth. The z component reaches a maximum value at the North pole, and is zero at the equator.

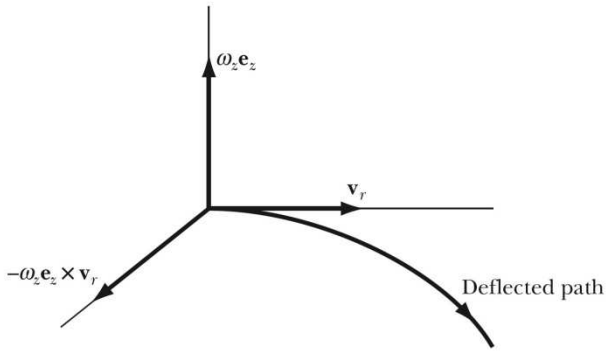


Figure 6. Deflection of a moving object as a result of the Coriolis force.

As a result of the Coriolis force, air flowing from West to East towards a region of low pressure will be deflected to the South on the Northern hemisphere. Air approaching the low from the East will be deflected to the North. On the Northern hemisphere we will thus expect that the air is flowing counter clockwise around an area of low pressure; in the same manner we can show that air flows clockwise around an area of high pressure. A lot about the weather can be understood on the basis of these observations. See for example the forecast map shown Figure 7. The position of the high pressure system over Michigan will bring cold air from Canada to Rochester (since the circulation around the high is in the clockwise direction). We thus expect the winds to be from the North. Once the high passes Rochester, the wind should come from the

South, bringing us higher temperatures. The low in the South of the USA will pull in moisture from the gulf of Mexico and rain and thunder can be expected in the region in front of the low (since this is the region where moisture of the golf of Mexico will go as a result of the counter-clockwise flow around the low).

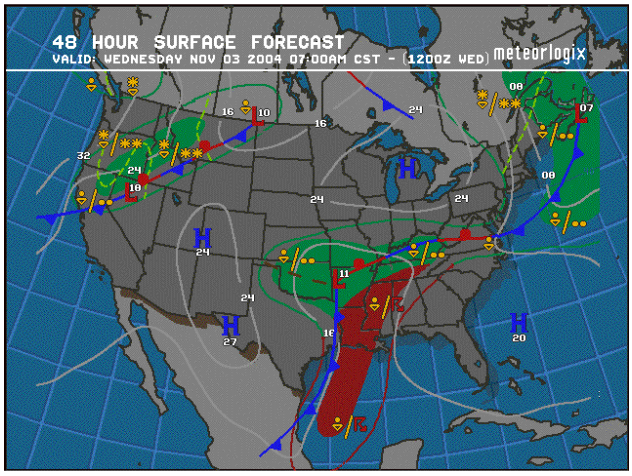


Figure 7. Forty-eight hour forecast map for Wednesday November 3, 2004, at 1200 Z (http://www.aopa.org/members/wx/focpage.cfm?sfcmap=0700d485).

Dynamics of Rigid Bodies

A rigid body is a collection of particles with fixed relative positions, independent of the motion carried out by the body. The dynamics of a rigid body has been discussed in our introductory courses, and the techniques discussed in these courses allow us to solve many problems in which the motion can be reduced to two-dimensional motion. In this special case, we found that the angular momentum associated with the rotation of the rigid object is directed in the same direction as the angular velocity:

$$\bar{L} = I\bar{\omega}$$

In this equation, I is the moment of inertia of the rigid body which was defined as

$$I = \sum_i m_i r_i^2$$

where r_i is the distance of mass m_i from the rotation axis. We also found that the kinetic energy of the body, associated with its rotation, is equal to

$$T = \frac{1}{2} I\omega^2$$

The complexity of the motion increases when we need three dimensions to describe the motion. There are many different ways to describe motion in three dimensions. One common method is to describe the motion of the center of mass (in a fixed coordinate system) and to describe the motion of the components around the center of mass (in the rotating coordinate system).

The Inertia Tensor

In Chapter 10 we derived the following relation between the velocity of a particle in the fixed reference frame, v_f, and its velocity in the rotating reference frame v_r:

$$\bar{v}_f = \bar{V} + \bar{v}_r + \bar{\omega} \times \bar{r}$$

If we assume that the rotating frame is fixed to the rigid body, then $v_r = 0$.

The total kinetic energy of the rigid body is the sum of the kinetic energies of each component of the rigid body. Thus

$$T = \sum_\alpha \left\{ \frac{1}{2} m_\alpha v_\alpha^2 \right\} = \frac{1}{2} \sum_\alpha m_\alpha \left(\left\{ \vec{V} + \vec{\omega} \times \vec{r}_\alpha \right\} \bullet \left\{ \vec{V} + \vec{\omega} \times \vec{r}_\alpha \right\} \right) =$$
$$= \frac{1}{2} \sum_\alpha m_\alpha \left(V^2 + 2\vec{V} \bullet \left\{ \vec{\omega} \times \vec{r}_\alpha \right\} + \left\{ \vec{\omega} \times \vec{r}_\alpha \right\} \bullet \left\{ \vec{\omega} \times \vec{r}_\alpha \right\} \right)$$

Let us now examine the three terms in this expression:

$$\frac{1}{2} \sum_\alpha m_\alpha V^2 = \frac{1}{2} V^2 \sum_\alpha m_\alpha = \frac{1}{2} M V^2$$

$$\frac{1}{2} \sum_\alpha m_\alpha \left(2\vec{V} \bullet \left\{ \vec{\omega} \times \vec{r}_\alpha \right\} \right) = \vec{V} \bullet \left\{ \vec{\omega} \times \sum_\alpha \left(m_\alpha \vec{r}_\alpha \right) \right\} = \vec{V} \bullet \left\{ \vec{\omega} \times \left(M \vec{R} \right) \right\} = 0$$

$$\frac{1}{2} \sum_\alpha m_\alpha \left(\left\{ \vec{\omega} \times \vec{r}_\alpha \right\} \bullet \left\{ \vec{\omega} \times \vec{r}_\alpha \right\} \right) = \frac{1}{2} \sum_\alpha m_\alpha \left(\omega^2 r_\alpha^2 - \left\{ \vec{\omega} \bullet \vec{r}_\alpha \right\}^2 \right) =$$
$$= \frac{1}{2} \sum_\alpha m_\alpha \left(\sum_i \omega_i^2 \sum_k r_{\alpha,k}^2 - \sum_i \left(\omega_i r_{\alpha,i} \right) \sum_j \left(\omega_j r_{\alpha,j} \right) \right) =$$
$$= \frac{1}{2} \sum_{i,j} \left\{ \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k r_{\alpha,k}^2 - r_{\alpha,i} r_{\alpha,j} \right) \right\} \omega_i \omega_j = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j$$

The second term is zero, if we choose the origin of the rotating coordinate system to coincide with the center of mass of the rigid object.

Using the previous expressions, we can now rewrite the total kinetic energy of the rigid object as

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j = T_{CM} + T_{rot}$$

The quantity I_{ij} is called the **inertia tensor**, and is a 3 x 3 matrix:

$$\{I\} = \begin{pmatrix} \sum_\alpha m_\alpha \left(r_{\alpha,2}^2 + r_{\alpha,3}^2 \right) & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,2} & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,1} & \sum_\alpha m_\alpha \left(r_{\alpha,1}^2 + r_{\alpha,3}^2 \right) & -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,1} & -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,2} & \sum_\alpha m_\alpha \left(r_{\alpha,1}^2 + r_{\alpha,2}^2 \right) \end{pmatrix}$$

Based on the definition of the inertia tensor we make the following observations:

- • The tensor is symmetric: $I_{ij} = I_{ji}$. Of the 9 parameters, only 6 are free parameters.
- • The non-diagonal tensor elements are called **products of inertia.**
- • The diagonal tensor elements are the moments of inertia with respect to the three coordinate axes of the rotating frame.

Angular Momentum

The total angular momentum L of the rotating rigid object is equal to the vector sum of the angular momenta of each component of the rigid object. The ith component of L is equal to

$$L_i = \sum_\alpha (\vec{r}_\alpha \times \vec{p}_\alpha)_i = \sum_\alpha (\vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha))_i = \sum_\alpha m_\alpha (\vec{r}_\alpha \times \vec{\omega} \times \vec{r}_\alpha)_i = \sum_\alpha m_\alpha (\vec{r}_\alpha^{\,2} \vec{\omega} - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega}))_i =$$

$$= \sum_\alpha m_\alpha \left\{ r_\alpha^2 \omega_i - r_{\alpha i} \sum_j (r_{\alpha j} \omega_j) \right\} = \sum_j \omega_j \sum_\alpha m_\alpha \{ r_\alpha^2 \delta_{ij} - r_{\alpha i} r_{\alpha j} \} = \sum_j I_{ij} \omega_j$$

This equation clearly shows that the angular momentum is in general not parallel to the angular velocity. An example of a system where the angular momentum is directed in a different from the angular velocity is shown in Figure 1.

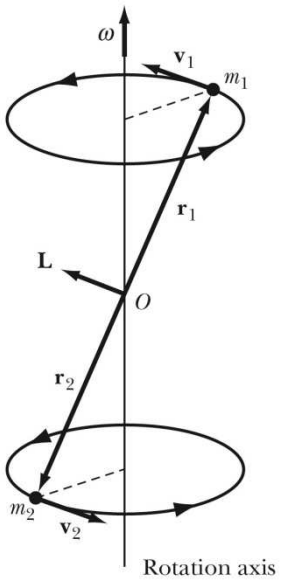


Figure 1. A rotating dumbbell is an example of a system in which the angular velocity is not parallel to the angular momentum.

The rotational kinetic energy can also be rewritten in terms of the angular momentum:

$$T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i \omega_i\left(\sum_j I_{ij}\omega_j\right) = \frac{1}{2}\sum_i \omega_i L_i = \frac{1}{2}(\bar{\omega}\bullet\bar{L})$$

Principal Axes

We always have the freedom to choose our coordinate axes such that the problem we are trying solve is simplified. When we are working on problems that involve the use of the inertia tensor, we can obtain a significant simplification if we can choose our coordinate axes such that the non-diagonal elements are 0. In this case, the inertia tensor would be equal to

$$\{I\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

For this inertia tensor we get the following relation between the angular momentum and the angular velocity:

$$L_i = I_i\omega_i$$

The rotational kinetic energy is equal to

$$T_{rot} = \frac{1}{2}\sum_i I_i\omega_i^2$$

The axes for which the non-diagonal matrix elements vanish are called the **principal axes of inertia**.

The biggest problem we are facing is how do we determine the proper coordinate axes? If the angular velocity vector is directed along one of the three coordinate axes that would get rid of the non-diagonal inertia tensor elements, we expect to see the following relation between the angular velocity vector and the angular momentum:

$$\bar{L} = I\bar{\omega}$$

Substituting the general form of the inertia tensor into this expression, we must require that

$$\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}$$

This set of equations can be rewritten as

$$\begin{aligned}(I_{11}-I)\omega_1+I_{12}\omega_2+I_{13}\omega_3&=0\\ I_{21}\omega_1+(I_{22}-I)\omega_2+I_{23}\omega_3&=0\\ I_{31}\omega_1+I_{32}\omega_2+(I_{33}-I)\omega_3&=0\end{aligned}$$

This set of equations only has a non-trivial solution if the determinant of the coefficients vanish. This requires that

$$\begin{vmatrix} I_{11}-I & I_{12} & I_{13} \\ I_{21} & I_{22}-I & I_{23} \\ I_{31} & I_{32} & I_{33}-I \end{vmatrix}=0$$

This requirement leads to three possible values of I. Each of these corresponds to the moment of inertia about one of the principal aces.

Transformations of the Inertia Tensor

In our discussion so far we have assumed that the origin of the rotating reference frame coincidence with the center of mass of the rigid object. In this Section we will examine what will change if we do not make this assumption.

Consider the two coordinate systems shown in Figure 2. One reference frame, the x frame, has its origin O coincide with the center of mass of the rigid object; the second reference frame, the X frame, has an origin Q that is displaced with respect to the center of mass of the rigid object.

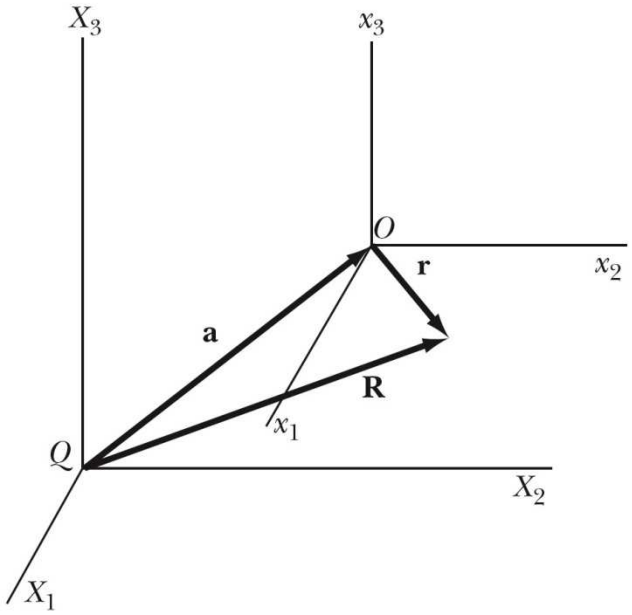


Figure 2. Two coordinate systems used to describe our rigid body.

The inertia tensor J_{ij} in reference frame X is defined in the same way as it was defined previously:

$$I_{ij} = \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k} X_{\alpha_i k}^{\ 2} - X_{\alpha_i i}X_{\alpha_i j}\right)$$

The coordinates in the X frame are related to the coordinates in the x frame in the following way:

$$X_i = a_i + x_i$$

Using this relation we can express the inertia tensor in reference frame X in terms of the coordinates in reference frame x:

$$\begin{aligned} I_{ij} &= \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k}\left(a_k + x_{\alpha_i k}\right)^2 - \left(a_i + x_{\alpha_i i}\right)\left(a_j + x_{\alpha_i j}\right)\right) = \\ &= \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k} x_{\alpha_i k}^{\ 2} - x_{\alpha_i i}x_{\alpha_i j}\right) + \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k}\left(a_k^2 + 2a_k x_{\alpha_i k}\right) - \left(a_i a_j + a_i x_{\alpha_i j} + a_j x_{\alpha_i i}\right)\right) = \\ &= I_{ij} + \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k} a_k^{\ 2} - a_i a_j\right) + \sum_{\alpha} m_{\alpha}\left(2\delta_{ij}\sum_{k}\left(2a_k x_{\alpha_i k}\right) - a_i x_{\alpha_i j} - a_j x_{\alpha_i i}\right) \end{aligned}$$

The last term on the right-hand side is equal to 0 since the origin of the coordinate system x coincides with the center of mass of the object:

$$\sum_{\alpha} m_{\alpha} x_{\alpha_i k} = M r_{cm} = 0$$

The relation between the inertia tensor in reference frame X and the inertia tensor in reference frame x is thus given by

$$I_{ij} = I_{ij} + \sum_{\alpha} m_{\alpha}\left(\delta_{ij}\sum_{k} a_k^{\ 2} - a_i a_j\right) = I_{ij} + M\left(\delta_{ij}a^2 - a_i a_j\right)$$

This relation is called the **Steiner's parallel-axis theorem** and is one example of how coordinate transformations affect the inertia tensor.

The transformation discussed so far is a simple translation. Other important transformations are rotations. In Chapter 1 we discussed many examples of rotations, and determined that the most general way to express rotations is by using the rotation matrix λ:

$$x_i = \sum_j \lambda_{ij} x'_j$$

Since this transformation rule is valid for vectors in general, the same rule can be used to describe the transformation of the angular momentum and angular velocity vectors:

$$L_i = \sum_j \lambda_{ij} L'_j$$

$$\omega_i = \sum_j \lambda_{ij} \omega'_j$$

In order to determine the relation between the inertia tensor in the two coordinate frames, we use the fact that the angular momentum is the product of the inertia tensor and the angular velocity, in both frames:

$$L_i = \sum_j I_{ij} \omega_j$$

and

$$L'_i = \sum_j I'_{ij} \omega'_j$$

In order to relate the inertia tensors, we use the coordinate transformations for L and ω:

$$\sum_j \lambda_{ij} L'_j = \sum_j I_{ij} \sum_m \lambda_{mj} \omega'_m$$

This equation can be simplified if we multiply each side by λ_{ik} and sum over k:

$$\sum_i \lambda_{ik} \left(\sum_j \lambda_{ij} L'_j \right) = \sum_j \left\{ \sum_i (\lambda_{ik} \lambda_{ij}) L'_j \right\} = \sum_j \{ \delta_{kj} L'_j \} = L'_k =$$

$$= \sum_i \lambda_{ik} \left(\sum_j I_{ij} \sum_m \lambda_{mj} \omega'_m \right) = \sum_m \left\{ \sum_{i,j} \lambda_{ik} \lambda_{mj} I_{ij} \right\} \omega'_m$$

where we have used the orthogonal properties of the rotation matrix. Using the relation between the angular momentum and the angular velocity in the rotated coordinate frame we see that the inertia tensors in the two coordinate frames are related as follows:

$$I'_{ma} = \sum_{b,i} \lambda_{ai} \lambda_{mb} I_{ib} = \sum_{b,i} \lambda_{ai} I_{ib} \lambda^t_{bm}$$

where λ^t is the transposed matrix. In tensor notation we can rewrite this relation as

$$\{I'\} = \{\lambda\}\{I\}\{\lambda^t\}$$

It turns out that for any inertia tensor we can find a rotation such that the inertia tensor in the rotated frame is a diagonal matrix (all non-diagonal elements are equal to 0).

We thus have seen two different approaches to diagonalize the inertia tensor: 1) find the principal axes of inertia, and 2) find the proper rotation matrix.

Euler Angles

Any rotation between different coordinate systems can be expressed in terms of three successive rotations around the coordinate axes. When we consider the transformation from the fixed coordinate system x' to the body coordinate system x, we call the three angles the Euler angles ϕ, θ, and ψ (see Figure 3).

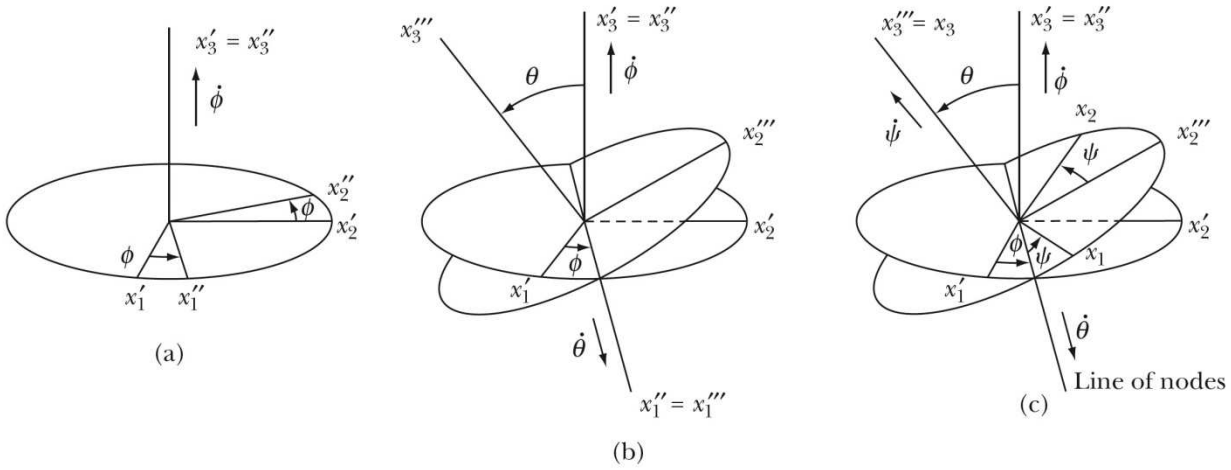


Figure 3. The Euler angles used to transform the fixed coordinate system x' into the body coordinate system x.

The total transformation matrix is the product of the individual transformations (note order)

$$\lambda = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

With each of the three rotations we can associate an angular velocity ω. To express the angular velocity in the body coordinate system, we can use Figure 3c.

- • ω_ϕ: Figure 3c shows that the angular velocity ω_ϕ is directed in the x_2''' - x_3''' plane. Its projection along the x_3''' axis, which is also the x_3 axis, is equal to

$$\dot{\phi}_3 = \dot{\phi}\cos\theta$$

The projection along the x_2''' axis is equal to

$$\dot{\phi}_{2'''} = \dot{\phi}\sin\theta$$

Figure 3c shows that when we project this projection along the x_1 and x_2 axes we obtain the following components in the body coordinate system:

$$\dot{\phi}_1 = \dot{\phi}_{2'''}\sin\psi = \dot{\phi}\sin\theta\sin\psi$$

$$\dot{\phi}_2 = \dot{\phi}_{2'''}\cos\psi = \dot{\phi}\sin\theta\cos\psi$$

- • ω_θ: Figure 3c shows that the angular velocity ω_θ is directed in the x_1''' - x_2''' plane. Its projection along the x_3''' axis, which is also the x_3 axis, is equal to 0.

$$\dot{\theta}_3 = 0$$

Figure 3c shows that when we project ω_θ along the x_1 and x_2 axes we obtain the following components in the body coordinate system:

$$\dot{\theta}_1 = \dot{\theta}\cos\psi$$

$$\dot{\theta}_2 = -\dot{\theta}\sin\psi$$

- • ω_ψ: Figure 3c shows that the angular velocity ω_ψ is directed along the x_3''' axis, which is also the x_3 axis. The components along the other body axes are 0. Thus:

$$\dot{\psi}_1 = 0$$

$$\dot{\psi}_2 = 0$$

$$\dot{\psi}_3 = \dot{\psi}$$

The angular velocity, in the body frame, is thus equal to

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 \\ \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 \\ \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

The Force-Free Euler Equations

Let's assume for the moment that the coordinate axis correspond to the principal axes of the body. In that case, we can write the kinetic energy of the body in the following manner:

$$T = \frac{1}{2}\sum_i I_i \omega_i^2$$

where I_i are the principal moments of the rigid body. If for now we consider that the rigid object is carrying out a force-free motion ($U = 0$) then the Lagrangian L will be equal to the kinetic energy T. The motion of the object can be described in terms of the Euler angles, which can serve as the generalized coordinates of the motion. Consider the three equations of motion for the three generalized coordinates:

- • The Euler angle ϕ: Lagrange's equation for the coordinate ϕ is

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \phi} - \frac{d}{dt}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\phi}} = \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \phi} - \frac{d}{dt}\sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\phi}} = 0$$

Differentiating the angular velocity with respect to the coordinate ϕ we find

$$\frac{\partial \omega_1}{\partial \phi} = 0 \qquad \frac{\partial \omega_1}{\partial \dot{\phi}} = 0$$

$$\frac{\partial \omega_2}{\partial \phi} = 0 \qquad \frac{\partial \omega_2}{\partial \dot{\phi}} = 0$$

$$\frac{\partial \omega_3}{\partial \phi} = 0 \qquad \frac{\partial \omega_3}{\partial \dot{\phi}} = \cos\theta$$

and Lagrange's equation becomes

$$\frac{d}{dt}\{I_{\cdot}\omega_{\cdot}\cos\theta\}=0$$

- • The Euler angle θ: Lagrange's equation for the coordinate θ is

$$0=\frac{\partial T}{\partial\theta}-\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}}=\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\theta}-\frac{d}{dt}\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\dot{\theta}}=\sum_i I_i\omega_i\frac{\partial\omega_i}{\partial\theta}-\frac{d}{dt}\sum_i I_i\omega_i\frac{\partial\omega_i}{\partial\dot{\theta}}=0$$

Differentiating the angular velocity with respect to the coordinate θ we find

$$\frac{\partial\omega_1}{\partial\theta}=\dot{\phi}\cos\theta\sin\psi \qquad \frac{\partial\omega_1}{\partial\dot{\theta}}=\cos\psi$$
$$\frac{\partial\omega_2}{\partial\theta}=\dot{\phi}\cos\theta\cos\psi \qquad \frac{\partial\omega_2}{\partial\dot{\theta}}=-\sin\psi$$
$$\frac{\partial\omega_3}{\partial\theta}=-\dot{\phi}\sin\theta \qquad \frac{\partial\omega_3}{\partial\dot{\theta}}=0$$

and Lagrange's equation becomes

$$\dot{\phi}\left(\{I_{\cdot}\omega_{\cdot}\sin\psi+I_{\_}\omega_{\_}\cos\psi\}\cos\theta-I_{\cdot}\omega_{\cdot}\sin\theta\right)-\frac{d}{dt}\{I_{\cdot}\omega_{\cdot}\cos\psi-I_{\_}\omega_{\_}\sin\psi\}=0$$

- • The Euler angle ψ: Lagrange's equation for the coordinate ψ is

$$0=\frac{\partial T}{\partial\psi}-\frac{d}{dt}\frac{\partial T}{\partial\dot{\psi}}=\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\psi}-\frac{d}{dt}\sum_i\frac{\partial T}{\partial\omega_i}\frac{\partial\omega_i}{\partial\dot{\psi}}=\sum_i I_i\omega_i\frac{\partial\omega_i}{\partial\psi}-\frac{d}{dt}\sum_i I_i\omega_i\frac{\partial\omega_i}{\partial\dot{\psi}}=0$$

Differentiating the angular velocity with respect to the coordinate ψ we find

$$\frac{\partial\omega_{\cdot}}{\partial\psi}=\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi=\omega_{\_} \qquad \frac{\partial\omega_{\cdot}}{\partial\dot{\theta}}=0$$
$$\frac{\partial\omega_{\_}}{\partial\psi}=-\dot{\phi}\sin\theta\sin\psi-\dot{\theta}\cos\psi=-\omega_{\cdot} \qquad \frac{\partial\omega_{\_}}{\partial\dot{\theta}}=0$$
$$\frac{\partial\omega_{\cdot}}{\partial\psi}=0 \qquad \frac{\partial\omega_{\cdot}}{\partial\dot{\theta}}=1$$

and Lagrange's equation becomes

$$I_1\omega_1\omega_2 - I_2\omega_2\omega_1 - \frac{d}{dt}\{I_3\omega_3\} = (I_1 - I_2)\omega_1\omega_2 - \frac{d}{dt}\{I_3\omega_3\} = (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

Of all three equations of motion, the last one is the only one to contain just the components of the angular velocity. Since our choice of the x_3 axis was arbitrary, we expect that similar relations should exist for the other two axes. The set of three equation we obtain in this way are called the **Euler equations**:

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0$$

As an example of how we use Euler's equations, consider a symmetric top. The top will have two different principal moments: $I_1 = I_2$ and I_3. In this case, the first Euler equations reduces to

$$I_3\dot{\omega}_3 = 0$$

or

$$\omega_3(t) = \text{constant} = \omega_3$$

The other two Euler equations can be rewritten as

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = -\Omega\omega_2$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 = \Omega\omega_1$$

This set of equations has the following solution:

$$\omega_1(t) = A\cos\Omega t$$

$$\omega_2(t) = A\sin\Omega t$$

The magnitude of the angular velocity of the system is constant since

$$|\vec{\omega}| = \sqrt{\omega_1^2(t) + \omega_2^2(t) + \omega_3^2} = \sqrt{\Omega^2 + \omega_3^2}$$

The angular velocity vector traces out a cone in the body frame (it precesses around the x_3 axis - see Figure 4). The rate with which the angular velocity vector precesses around the x_3 axis is determined by the value of Ω. When the principal moment I_3 and the principal moment I_1 are similar, Ω will become very small.

Since we have assumed that there are no external forces and torques acting on the system, the angular momentum of the system will be constant in the fixed reference frame. If the angular momentum is initially pointing along the x'_3 axis it will continue to point along this axis (see Figure 5). Since there are no external forces and torques acting on the system, the rotation kinetic energy of the system must be constant. Thus

$$T_{rot} = \frac{1}{2}\bar{\omega} \cdot \bar{L}$$

Since the angle between the angular velocity vector and the angular momentum vector must be constant, the angular velocity vector must trace out a space cone around the x'_3 (see Figure 5).

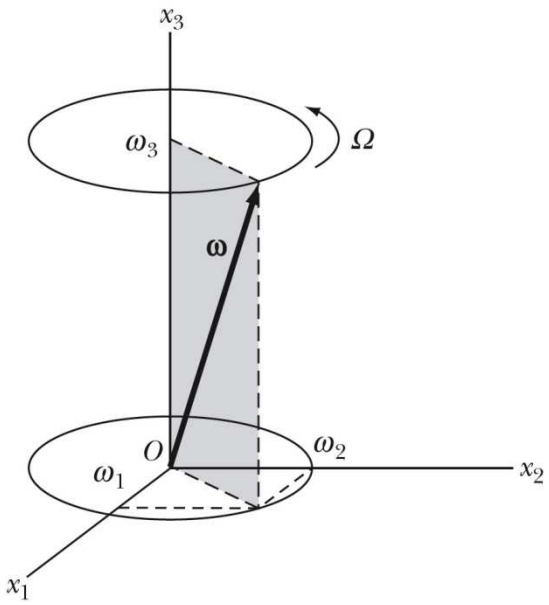


Figure 4. The angular velocity of a force-free symmetric top, precessing around the x_3 axis in the body frame.

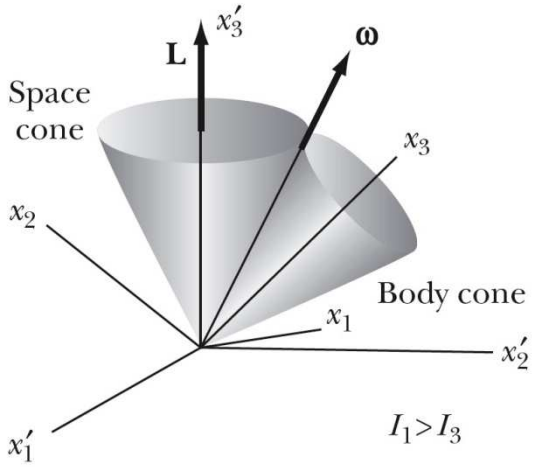


Figure 5. The angular velocity of a force-free symmetric top, tracing out a space-cone around the x'_3 axis in the body frame.

The Euler Equations in a Force Field

When the external forces and torques acting on the system are not equal to 0, we can not use the method we have used in the previous section to obtain expressions for the angular velocity and acceleration. The procedure used in the previous section relied on the fact that the potential

energy U is 0 in a force-free environment, and therefore, the Lagrangian L is equal to the kinetic energy T.

When the external forces and torques are not equal to 0, the angular momentum of the system is not conserved:

$$\left(\frac{d\bar{L}}{dt}\right)_{fixed} = \bar{N}$$

Note that this relation only holds in the fixed reference frame since this is the only good inertial reference frame. In Chapter 10 we looked at the relation between parameters specified in the fixed reference frame compared to parameters specified in the rotating reference frame, and we can use this relation to correlate the rate of change of the angular momentum vector in the fixed reference frame with the rate of change of the angular momentum vector in the rotating reference frame:

$$\bar{N} = \left(\frac{d\bar{L}}{dt}\right)_{fixed} = \left(\frac{d\bar{L}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{L}$$

This relation can be used to generate three separate relations by projecting the vectors along the three body axes:

$$N_1 = \frac{dL_1}{dt} + (\bar{\omega} \times \bar{L})_1 = \frac{dL_1}{dt} + (\omega_2 L_3 - \omega_3 L_2) = I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3$$

$$N_2 = \frac{dL_2}{dt} + (\bar{\omega} \times \bar{L})_2 = \frac{dL_2}{dt} + (\omega_3 L_1 - \omega_1 L_3) = I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1$$

$$N_3 = \frac{dL_3}{dt} + (\bar{\omega} \times \bar{L})_3 = \frac{dL_3}{dt} + (\omega_1 L_2 - \omega_2 L_1) = I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2$$

These equations are the Euler equations for the motion of the rigid body in a force field. In the absence of a torque, these equations reduce to the force-free Euler equations.

Example: Motion of a Symmetric Top with One Point Fixed

In order to describe the motion of a top, which has its tip fixed, we use two coordinate systems whose origins coincide (see Figure 6). Since the origins coincide, the transformation between coordinate systems can be described in terms of the Euler angles, and the equations of motion will be the Euler equations:

$$(I_1 - I_2)\omega_1 \omega_2 - I_3 \dot{\omega}_3 = 0$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0$$

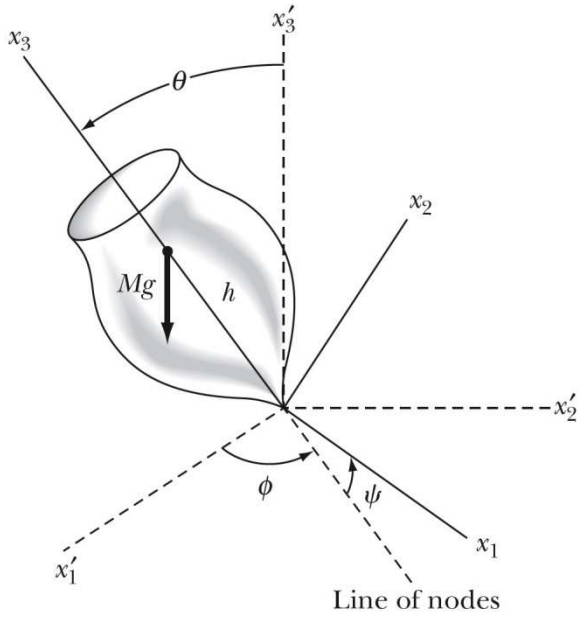


Figure 6. Spinning top with fixed tip.

Since the top is symmetric around the x_3 axis, it principal moments of inertia with respect to the x_1 and x_2 axes are identical. The Euler equations now become

$$I_3\dot{\omega}_3 = 0$$

$$(I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$-(I_1 - I_3)\omega_3\omega_1 - I_1\dot{\omega}_2 = 0$$

The first equation immediately tells us that

$$\omega_3 = \text{constant}$$

The motion of the top is often described in terms of the motion of its rotating axes. The kinetic energy of the system is equal to

$$T = \frac{1}{2}\sum_i I_i\omega_i^2 = \frac{1}{2}I_1\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

The potential energy of the system, assuming the center of mass of the top is located a distance h from the tip, is equal to

$$U = Mgh\cos\theta$$

The Lagrangian is thus equal to

$$L = T - U = \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgh\cos\theta$$

The Lagrangian does not depend on ϕ and ψ, and thus

$$\frac{\partial L}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$$

$$\frac{\partial L}{\partial \psi} = 0 = \frac{\partial L}{\partial \dot{\psi}}$$

We thus conclude that the angular momenta associated with the Euler angles ϕ and ψ are constant:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{constant}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = I_3\omega_3 = \text{constant}$$

Expressing the momenta in terms of the Euler angles ϕ and ψ allows us to express the rate of change of these Euler angles in terms of the angular momenta:

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

$$\dot{\psi} = \frac{\left(p_\phi - p_\psi\cos\theta\right)\cos\theta}{I_1\sin^2\theta} = \dot{\phi}\cos\theta$$

Since there are no non-conservative forces acting on the top, the total energy E of the system is conserved. Thus

$$E = T + U = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 + Mgh\cos\theta = \text{constant}$$

The total energy can be rewritten in terms of the angular momenta:

$$\begin{aligned} E &= \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 + Mgh\cos\theta = \\ &= \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \omega_3^2 + Mgh\cos\theta = \\ &= \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2 \theta + \frac{1}{2} I_3 \omega_3^2 + Mgh\cos\theta = \\ &= \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{\left(p_\phi - p_\psi \cos\theta \right)^2}{I_1 \sin^2 \theta} + \frac{1}{2} I_3 \omega_3^2 + Mgh\cos\theta \end{aligned}$$

Since the angular velocity with respect to the $x3$ axis is constant, we can subtract it from the energy E to get the effective energy E' (note: this is equivalent to choosing the zero point of the energy scale). Thus

$$E' = E - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{\left(p_\phi - p_\psi \cos\theta \right)^2}{I_1 \sin^2 \theta} + Mgh\cos\theta = \text{constant}$$

The effective energy only depends on the angle θ and on $d\theta/dt$ since the angular momenta are constants. The manipulations we have carried out have reduced the three-dimensional problem to a one-dimensional problem. The first term in the effective energy is the kinetic energy associated with the rotation around the x_1 axis. The last two terms depend only on the angle θ and not on the angular velocity $d\theta/dt$. These terms are what we could call the effective potential energy, defined as

$$V(\theta) = \frac{1}{2} \frac{\left(p_\phi - p_\psi \cos\theta \right)^2}{I_1 \sin^2 \theta} + Mgh\cos\theta$$

The effective potential becomes large when the angle approaches 0 and π. The angular dependence of the effective potential is shown in Figure 7. If the total effective energy of the system is E_1', we expect the angle θ to vary between θ_1 and θ_2. We thus expect that the angle of inclination of the top will vary between these two extremes.

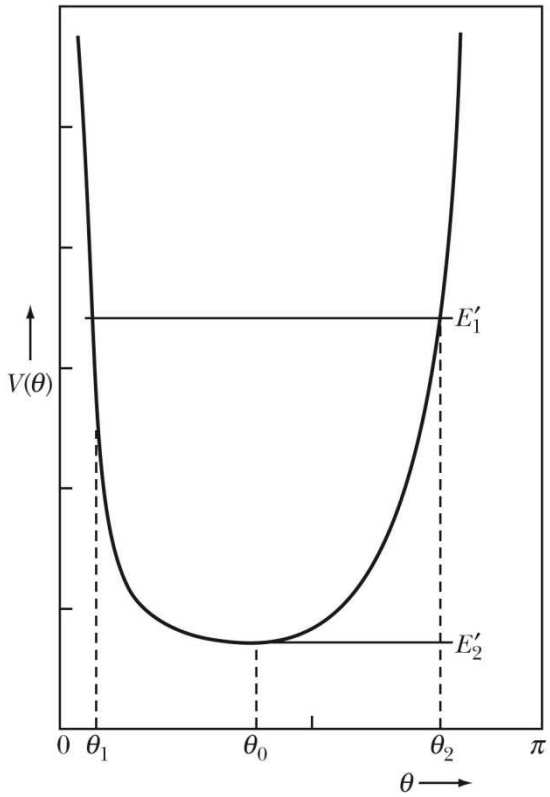


Figure 7. The effective potential of a rotating top.

The minimum effective energy that the system can have is E_2'. The corresponding angle can be found by requiring

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = 0$$

This requirement can be rewritten as a quadratic equation of a parameter β, where β is defined as

$$\beta = p_\theta - p_\varphi \cos\theta_0$$

In general, there are two solutions to this quadratic equation. Since β is a real number, the solution must be real, and this requires that

$$1 - \frac{4MghI_\perp \cos\theta_0}{p_\varphi^2} \geq 0$$

This equation can be rewritten as

$$p_\varphi^2 = (I_\perp \omega_\perp)^2 \geq 4MghI_\perp \cos\theta_0$$

When we study a spinning top, the spin axis is oriented such that $\theta_0 < \pi/2$. The previous equation can then be rewritten as

$$I_3\omega_3 \geq \sqrt{4MghI_1\cos\theta_0}$$

or

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0}$$

There is thus a minimum angular velocity the system must have in order to produce stable precession. The rate of precession can be found by calculating

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta_0}{I_1\sin^2\theta_0} = \frac{\beta}{I_1\sin^2\theta_0}$$

Since β has two possible values, we expect to see two different precession rates: one resulting in fast precession, and one resulting in slow precession.

When the angle of inclination is not equal to θ_0, the system will oscillate between two limiting values of θ. The precession rate will be a function of θ and can vary between positive and negative values, depending on the values of the angular momenta. The phenomenon is called nutation, and possible nutation patterns are shown in Figure 8. The type of nutation depends on the initial conditions.

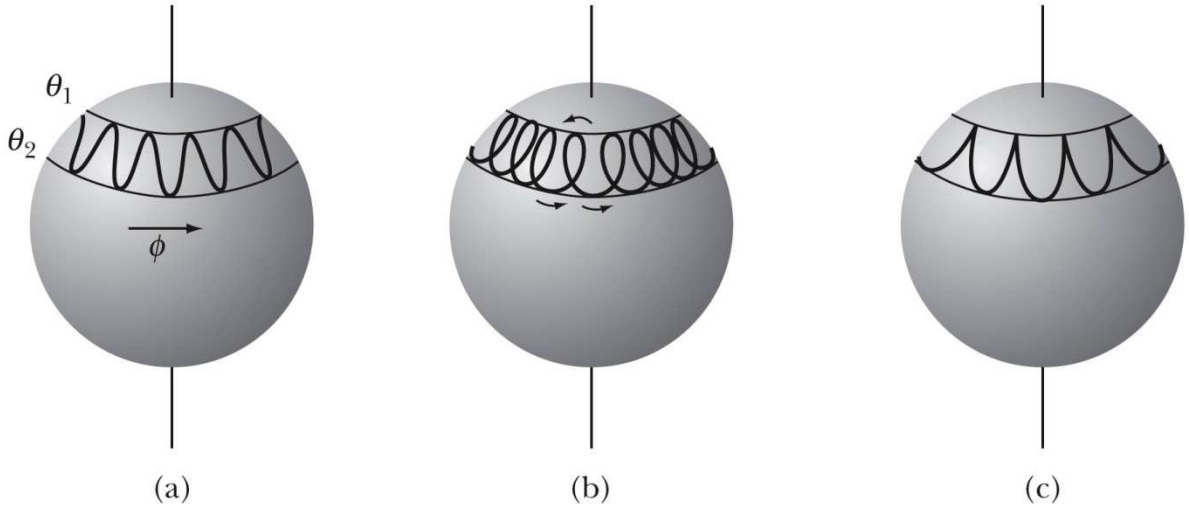


Figure 8. The nutation of a rotating top.

Stability of Rigid-Body Rotations

The rotation of a rigid body is stable if the system, when perturbed from its equilibrium condition, carries out small oscillations about it. Consider we use the principal axes of rotation to describe the motion, and we choose these axes such that $I_3 > I_2 > I_1$. If the system rotates around the x_1 axis we can write the angular velocity vector as

$$\bar{\omega} = \omega_1 \hat{x}_1$$

Consider what happens when we apply a small perturbation around the other two principal axes such that the angular velocity becomes

$$\bar{\omega} = \omega_1 \hat{x}_1 + \lambda \hat{x}_2 + \mu \hat{x}_3$$

The corresponding Euler equations are

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = (I_2 - I_3)\lambda\omega_1 - I_1\dot{\mu} = 0$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = (I_3 - I_1)\lambda\mu - I_2\dot{\omega}_2 = 0$$

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = (I_1 - I_2)\mu\omega_1 - I_3\dot{\lambda} = 0$$

Since we are talking about small perturbations from the equilibrium state, $\lambda\mu$ will be small and can be set to 0. The second equation can thus be used to conclude that

$$\omega_1 = \text{constant}$$

The remaining equations can be rewritten as

$$\dot{\mu} = \left(\frac{I_2 - I_3}{I_1}\omega_1\right)\lambda$$

$$\dot{\lambda} = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\mu$$

The last equation can be differentiated to obtain

$$\ddot{\lambda} = \frac{d\dot{\lambda}}{dt} = \left(\frac{I_2 - I_3}{I_1}\omega_3\right)\frac{d\mu}{dt} = \left(\frac{I_2 - I_3}{I_1}\omega_3\right)\left(\frac{I_3 - I_1}{I_2}\omega_3\right)\lambda = -\left(\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\omega_3^2\right)\lambda$$

The term within the parenthesis is positive since we assumed that $I_3 > I_2 > I_1$. This differential equation has the following solution:

$$\lambda(t) = A_\lambda \cos(\Omega_1 t) + B_\lambda \sin(\Omega_1 t)$$

where

$$\Omega_1 = \omega_3\sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_2 I_3}}$$

When we look at the perturbation around the x_3 axis we find the following differential equation

$$\ddot{\mu} = \frac{d\dot{\mu}}{dt} = \left(\frac{I_3 - I_1}{I_2}\omega_3\right)\frac{d\lambda}{dt} = \left(\frac{I_3 - I_1}{I_2}\omega_3\right)\left(\frac{I_2 - I_3}{I_1}\omega_3\right)\mu = -\left(\frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2}\omega_3^2\right)\mu$$

The solution of the second-order differential equation is

$$\mu(t) = A_\mu \cos(\Omega_1 t) + B_\mu \sin(\Omega_1 t)$$

We see that the perturbations around the x_2 axis and the x_3 axis oscillate around the equilibrium values of $\lambda = \mu = 0$. We thus conclude that the rotation around the x_1 axis is stable.

Similar calculations can be done for rotations around the x_2 axis and the x_3 axis. The perturbation frequencies obtained in those cases are equal to

$$\Omega_2 = \omega_2\sqrt{\frac{(I_1 - I_2)(I_3 - I_2)}{I_3 I_1}}$$

$$\Omega_3 = \omega_3\sqrt{\frac{(I_1 - I_3)(I_2 - I_3)}{I_2 I_1}}$$

We see that the first frequency is an imaginary number while the second frequency is a real number. Thus, the rotation around the x_3 axis is stable, but the rotation around the x_2 axis is unstable.

Coupled Oscillations

Many important physics systems involved coupled oscillators. Coupled oscillators are oscillators connected in such a way that energy can be transferred between them. The motion of coupled oscillators can be complex, and does not have to be periodic. However, when the oscillators carry out complex motion, we can find a coordinate frame in which each oscillator oscillates with a very well defined frequency.

A solid is a good example of a system that can be described in terms of coupled oscillations. The atoms oscillate around their equilibrium positions, and the interaction between the atoms is responsible for the coupling. To start our study of coupled oscillations, we will assume that the forces involved are spring-like forces (the magnitude of the force is proportional to the magnitude of the displacement from equilibrium).

Two Coupled Harmonic Oscillators

Consider a system of two objects of mass M. The two objects are attached to two springs with spring constants κ (see Figure 1). The interaction force between the masses is represented by a third spring with spring constant κ_{12}, which connects the two masses.

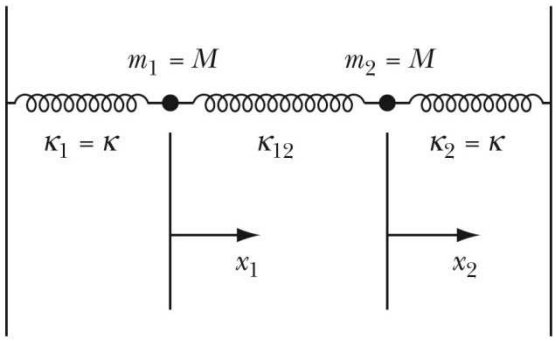


Figure 1. Two coupled harmonic oscillators.

We will assume that when the masses are in their equilibrium position, the springs are also in their equilibrium positions. The force on the left mass is equal to

$$F_1 = -\kappa x_1 + \kappa_{12}(x_2 - x_1) = -(\kappa + \kappa_{12})x_1 + \kappa_{12}x_2 = M\ddot{x}_1$$

The force on the right mass is equal to

$$F_2 = -\kappa x_2 + \kappa_{12}(x_1 - x_2) = -(\kappa + \kappa_{12})x_2 + \kappa_{12}x_1 = M\ddot{x}_2$$

The equations of motion are thus

$$M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0$$

$$M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0$$

Since it is reasonable to assume that the resulting motion has an oscillatory behavior, we consider following trial functions:

$$x_1(t) = B_1 e^{i\omega t}$$

$$x_2(t) = B_2 e^{i\omega t}$$

Substituting these trial functions into the equations of motion we obtain the following conditions:

$$(\kappa + \kappa_{12} - M\omega^2)B_1 - \kappa_{12}B_2 = 0$$

$$-\kappa_{12}B_1 + (\kappa + \kappa_{12} - M\omega^2)B_2 = 0$$

These equations only will have a non-trivial solution if

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0$$

Note: the trivial solution is $B_1 = B_2 = 0$. The requirement for a non-trivial solution requires that the angular frequency of the system is equal to one of the following two characteristic frequencies (the so called eigen frequencies):

$$\omega_+ = \pm\sqrt{\frac{\kappa + 2\kappa_{12}}{M}}$$

$$\omega_- = \pm\sqrt{\frac{\kappa}{M}}$$

For each of these frequencies, we can now determine the amplitudes B_1 and B_2. Let us first consider the eigen frequency ω_1. For this frequency we obtain the following relations between B_1 and B_2:

$$(\kappa + \kappa_{12} - (\kappa + 2\kappa_{12}))B_1 - \kappa_{12}B_2 = -\kappa_{12}B_1 - \kappa_{12}B_2 = -\kappa_{12}(B_1 + B_2) = 0$$

or $B_1 = -B_2$. For the eigen frequency ω_2 we obtain the following relations between B_1 and B_2:

$$(\kappa+\kappa_{12}-\kappa)B_1-\kappa_{12}B_2=\kappa_{12}B_1-\kappa_{12}B_2=\kappa_{12}(B_1-B_2)=0$$

or $B_1 = B_2$. The most general solution of the coupled harmonic oscillator problem is thus

$$x_1(t)=B_1^+e^{+i\omega_+t}+B_1^-e^{-i\omega_+t}+B_2^+e^{+i\omega_-t}+B_2^-e^{-i\omega_-t}$$

$$x_2(t)=-B_1^+e^{+i\omega_+t}-B_1^-e^{-i\omega_+t}+B_2^+e^{+i\omega_-t}+B_2^-e^{-i\omega_-t}$$

Another approach that can be used to solve the coupled harmonic oscillator problem is to carry out a coordinate transformation that decouples the coupled equations. Consider the two equations of motion. If we add them together we get

$$M(\ddot{x}_+ + \ddot{x}_-)+\kappa(x_+ + x_-)=0$$

If we subtract from each other we get

$$M(\ddot{x}_1-\ddot{x}_2)+(\kappa+2\kappa_{12})(x_1-x_2)=0$$

Based on these two equations it is clear that in order to decouple the equations of motion we need to introduce the following variables

$$\eta_1=x_1-x_2$$

$$\eta_2=x_1+x_2$$

The solutions to the decoupled equations of motion are

$$\eta_+(t)=C_+^+e^{i\omega_+t}+C_+^-e^{-i\omega_+t}$$

$$\eta_2(t)=C_2^+e^{i\omega_-t}+C_2^-e^{-i\omega_-t}$$

where the frequencies are the characteristic frequencies discussed before. Once we have these solutions we can determine the positions of the masses as function of time:

$$x_1(t)=\frac{1}{2}(\eta_2(t)+\eta_1(t))$$

$$x_-(t)=\frac{1}{2}(\eta_-(t)-\eta_+(t))$$

We note that the solution η_1 corresponds to an asymmetric motion of the masses, while the solution η_2 corresponds to an asymmetric motion of the masses (see Figure 2). Since higher frequencies correspond to higher energies, the asymmetric mode (out of phase) has a higher energy.

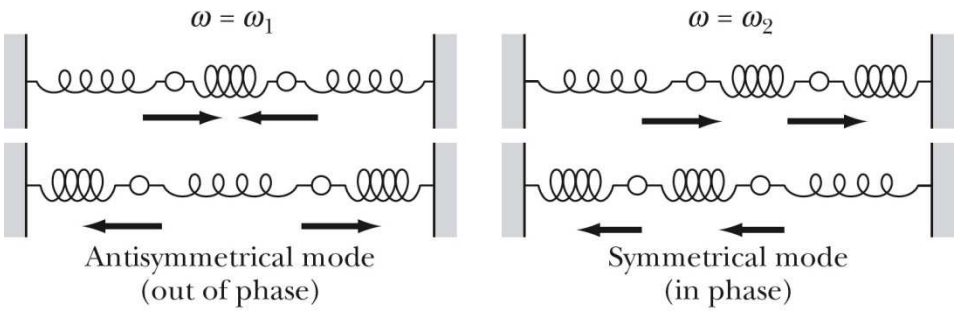


Figure 2. Normal modes of oscillation.

Weak Coupling

Coupled oscillations, involving a weak coupling, are important to describe many physical systems. For example, in many solids, the force that tie the atoms to their equilibrium positions are very much stronger than the inter-atomic coupling forces. In the example we discussed in the pervious section, the weak coupling limit requires that $\kappa_{12} << \kappa$. In this approximation we can show (see text book for details) that our solutions have a high-frequency component that oscillates inside a slowly varying component (see Figure 3). The solutions are thus sinusoidal functions with a slowly varying amplitude.

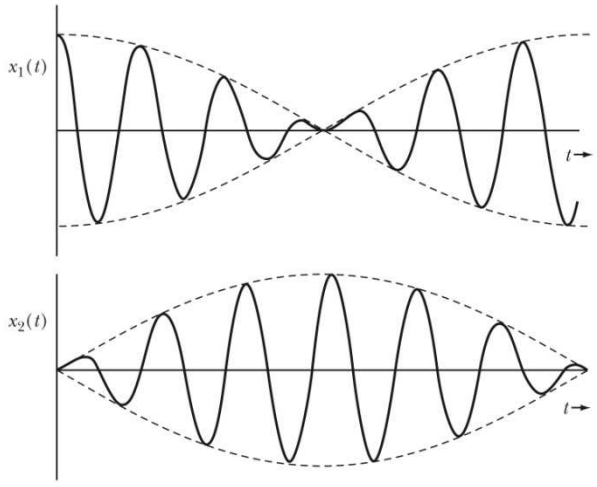


Figure 3. Examples of solution in the weak-coupling limit.

General Problem of Coupled Oscillations

The results of our study of the coupled harmonic oscillator problem results in a number of different observations:

- ○ The coupling in a system with two degrees of freedom results in two characteristic frequencies.
- ○ The two characteristic frequencies in a system with two degree of freedom are pushed towards lower and higher energies compared to the non-coupled frequency.

Let us now consider a system with *n* coupled oscillators. We can describe the state of this system in terms of *n* generalized coordinates q_i. The configuration of the system will be described with respect to the equilibrium state of the system (at equilibrium, the generalized coordinates are 0, and the generalized velocity and acceleration are 0). The evolution of the system can be described using Lagrange's equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

The second term on the left-hand side will contain terms that include the generalized velocity and the generalized acceleration, and is thus equal to 0 at the equilibrium position. Lagrange's equations thus tells us that

$$\left.\frac{\partial L}{\partial q_i}\right|_0 = 0 = \left.\frac{\partial T}{\partial q_i}\right|_0 - \left.\frac{\partial U}{\partial q_i}\right|_0$$

However, since we know how to express the kinetic energy of the system in terms of the generalized coordinates we conclude that

$$\frac{\partial T}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k\right) = 0$$

where

$$m_{jk} = \sum_\alpha m_\alpha \sum_i \left(\frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}\right)$$

For the potential energy *U* we conclude that

$$\left.\frac{\partial U}{\partial q_i}\right|_0 = \left.\frac{\partial T}{\partial q_i}\right|_0 = 0$$

The potential energy can be expanded around the equilibrium position using a Taylor series and we find that

$$U(q_1, q_2, \ldots) = U_0 + \sum_k \left(\frac{\partial U}{\partial q_k} \Big|_0 \right) q_k + \frac{1}{2} \sum_{i,k} \left(\frac{\partial^2 U}{\partial q_i \partial q_k} \Big|_0 \right) q_i q_k + \ldots \approx \frac{1}{2} \sum_{i,k} \left(\frac{\partial^2 U}{\partial q_i \partial q_k} \Big|_0 \right) q_i q_k = \frac{1}{2} \sum_{i,k} A_{ik} q_i q_k$$

where

$$A_{ik} = \frac{\partial^2 U}{\partial q_i \partial q_k} \Big|_0$$

We thus conclude that:

$$\frac{\partial L}{\partial q_i} = \frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} = -\frac{\partial U}{\partial q_i} = -\frac{1}{2} \sum_k A_{ki} q_k$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} \sum_k m_{ki} \dot{q}_k$$

The equation of motion can now be written as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\frac{1}{2} \sum_k A_{ki} q_k - \frac{d}{dt} \left(\frac{1}{2} \sum_k m_{ki} \dot{q}_k \right) = -\frac{1}{2} \sum_k \left(A_{ki} q_k + m_{ki} \ddot{q}_k \right) = 0$$

The index *k* runs over all degrees of freedom of the system, and we thus have *n* second order differential equations. In order to find the general solution we try a trial solution that exhibits the expected oscillatory behavior:

$$q_i(t) = a_i e^{i(\omega t - \delta)}$$

With this solution, the equations of motion become

$$\sum_k \left(A_{ki} - \omega^2 m_{ki} \right) a_k = 0$$

This set of equations will only has non-trivial solutions if the determinant of the coefficient must vanish:

$$\begin{vmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} & A_{13} - \omega^2 m_{13} & \ldots \\ A_{21} - \omega^2 m_{21} & A_{22} - \omega^2 m_{22} & A_{23} - \omega^2 m_{23} & \ldots \\ A_{31} - \omega^2 m_{31} & A_{32} - \omega^2 m_{32} & A_{33} - \omega^2 m_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

In general there will be n different values of the angular frequency. These frequencies are called the characteristic frequencies or eigen frequencies. Depending on the coefficients, some of the characteristic frequencies are the same (this phenomena is called degeneracy). For each eigen frequency we can determine the ratio of the amplitudes; these amplitudes define an n-dimensional vector, also called the eigen vector. Note: the eigen vector has a pure harmonic time dependence.

The general solution of the system is a linear combination of the solutions q_i. Of course, it is only the real part of the solutions that is meaningful.

The normal coordinates can be determined by finding the appropriate linear combinations of solutions q_i that oscillates at a single frequency. These normal coordinates are

$$\eta_r(t) = \beta_r e^{i\omega_r t}$$

The amplitude may be a complex number. The normal coordinates must satisfy the following relation

$$\ddot{\eta}_r + \omega^2 \eta_r = 0$$

Since there are n equations of motion, we also expect to see n normal coordinates, and n decoupled equations of motion.

To illustrate the detailed steps to be followed to solve a coupled oscillator problem we will examine Example 12.4 from the textbook. In this example, the coupled pendulum shown in Figure 4 is examined.

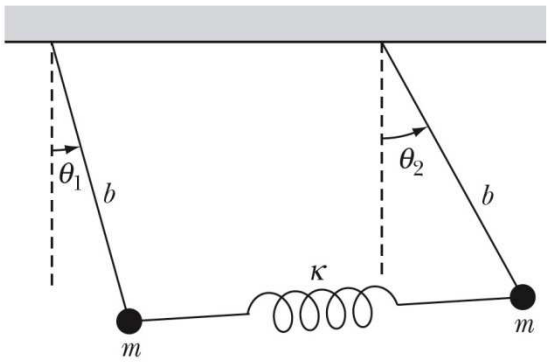


Figure 4. Coupled pendulum of Example 12.4.

1. **Choose generalized coordinates**. The proper generalized coordinates in this problem are the angles θ_1 and θ_2. The kinetic and the potential energy of the system can be easily expressed in terms of these angles. We make the assumption that the spring is massless and there is thus no kinetic energy associated with the motion of the spring. The kinetic energy of the system is thus just equal to the kinetic energy of the two masses, and thus equal to

$$T = \frac{1}{2}m\left(b\dot{\theta}_1\right)^2 + \frac{1}{2}m\left(b\dot{\theta}_2\right)^2 = \frac{1}{2}mb^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

The potential energy of the system is the sum of the potential energy associated with the change in the height of the masses and the potential energy associated with the extension or compression of the spring. The total potential energy is thus equal to

$$\begin{aligned} U &= mg(b - b\cos\theta_1) + mg(b - b\cos\theta_2) + \frac{1}{2}\kappa\left(b\sin\theta_1 - b\sin\theta_2\right)^2 = \\ &= mgb\left(1 - \left(1 - \frac{1}{2}\theta_1^2\right)\right) + mgb\left(1 - \left(1 - \frac{1}{2}\theta_2^2\right)\right) + \frac{1}{2}\kappa b^2\left(\theta_1 - \theta_2\right)^2 = \\ &= \frac{1}{2}mgb\left(\theta_1^2 + \theta_2^2\right) + \frac{1}{2}\kappa b^2\left(\theta_1 - \theta_2\right)^2 = \frac{1}{2}mgb\left(\theta_1^2 + \theta_2^2\right) + \frac{1}{2}\kappa b^2\left(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2\right) \end{aligned}$$

We have used the small angle approximation in order to express the sin and cos of the angles in terms of the angles.

2. **Determine the *A* and *m* tensors**. In order to calculate these tensors we use the expressions for *T* and *U* obtained in step 1. Since the kinetic energy obtained in step 1 does not contain products of the generalized velocity of mass 1 and the generalized velocity of mass 2, the mass tensor will be a diagonal tensor. We can see this by looking at the definition of the mass tensor elements:

$$m_{jk} = \sum_\alpha m_\alpha \sum_i \left(\frac{\partial x_{\alpha i}}{\partial q_j} \frac{\partial x_{\alpha i}}{\partial q_k} \right)\Bigg|_0 = mb^2 \left\{ \frac{\partial\theta_1}{\partial\theta_j}\frac{\partial\theta_1}{\partial\theta_k} + \frac{1}{4}\frac{\partial\theta_1^2}{\partial\theta_j}\frac{\partial\theta_1^2}{\partial\theta_k} + \frac{\partial\theta_2}{\partial\theta_j}\frac{\partial\theta_2}{\partial\theta_k} + \frac{1}{4}\frac{\partial\theta_2^2}{\partial\theta_j}\frac{\partial\theta_2^2}{\partial\theta_k} \right\}\Bigg|_0$$

The mass tensor is thus equal to

$$m = \begin{Bmatrix} mb^2 & 0 \\ 0 & mb^2 \end{Bmatrix}$$

The *A* tensor is equal to

$$A = \begin{Bmatrix} \dfrac{\partial^2 U}{\partial\theta_1\partial\theta_1}\Big|_0 = \left(mgb + \kappa b^2\right)\Big|_0 = mgb + \kappa b^2 & \dfrac{\partial^2 U}{\partial\theta_1\partial\theta_2}\Big|_0 = \left(-\kappa b^2\right)\Big|_0 = -\kappa b^2 \\ \dfrac{\partial^2 U}{\partial\theta_2\partial\theta_1}\Big|_0 = \left(-\kappa b^2\right)\Big|_0 = -\kappa b^2 & \dfrac{\partial^2 U}{\partial\theta_2\partial\theta_2}\Big|_0 = \left(mgb + \kappa b^2\right)\Big|_0 = mgb + \kappa b^2 \end{Bmatrix}$$

3. **Determine the eigen frequency and the eigen vectors**. The eigen frequencies can be determined by requiring that the determinant of the coefficients of the equations of motions vanishes:

$$\left|\{A\}-\{m\}\omega^2\right| = \left|\begin{pmatrix} mgb+\kappa b^2 & -\kappa b^2 \\ -\kappa b^2 & mgb+\kappa b^2 \end{pmatrix}-\begin{pmatrix} mb^2 & 0 \\ 0 & mb^2 \end{pmatrix}\omega^2\right| =$$

$$= \begin{vmatrix} mgb+\kappa b^2-mb^2\omega^2 & -\kappa b^2 \\ -\kappa b^2 & mgb+\kappa b^2-mb^2\omega^2 \end{vmatrix} =$$

$$= mb^2\begin{vmatrix} \frac{g}{b}+\frac{\kappa}{m}-\omega^2 & -\frac{\kappa}{m} \\ -\frac{\kappa}{m} & \frac{g}{b}+\frac{\kappa}{m}-\omega^2 \end{vmatrix} = 0$$

This requires that

$$\left(\frac{g}{b}+\frac{\kappa}{m}-\omega^2\right)^2-\left(\frac{\kappa}{m}\right)^2=0$$

or

$$\frac{g}{b}+\frac{\kappa}{m}-\omega^2=\pm\frac{\kappa}{m}$$

The eigen frequencies are thus equal to

$$\omega=\sqrt{\frac{g}{b}+\frac{\kappa}{m}\pm\frac{\kappa}{m}}=\begin{cases}\sqrt{\frac{g}{b}} \\ \sqrt{\frac{g}{b}+2\frac{\kappa}{m}}\end{cases}$$

Consider the first eigen frequency. For this frequency, the eigen vector is (a_{11}, a_{21}). The equations of motion for this frequency are

$$\left(\frac{g}{b}+\frac{\kappa}{m}-\omega^2\right)a_{11}-\frac{\kappa}{m}a_{21}=\frac{\kappa}{m}a_{11}-\frac{\kappa}{m}a_{21}=\frac{\kappa}{m}(a_{11}-a_{21})=0$$
$$-\frac{\kappa}{m}a_{11}+\left(\frac{g}{b}+\frac{\kappa}{m}-\omega^2\right)a_{21}=-\frac{\kappa}{m}a_{11}+\frac{\kappa}{m}a_{21}=-\frac{\kappa}{m}(a_{11}-a_{21})=0$$

Each of these two equations tells us that $a_{11} = a_{21}$.

Since the eigen vectors are orthogonal, we expect that the eigen vector for the second eigen frequency is given by $a_{12} = -a_{22}$. We come to the same conclusion if we start from the equations of motion for that frequency and the eigen vector (a_{12}, a_{22}):

$$\left(\frac{g}{b}+\frac{\kappa}{m}-\omega^2\right)a_{12}-\frac{\kappa}{m}a_{22}=-\frac{\kappa}{m}a_{12}-\frac{\kappa}{m}a_{22}=-\frac{\kappa}{m}(a_{12}+a_{22})=0$$
$$-\frac{\kappa}{m}a_{11}+\left(\frac{g}{b}+\frac{\kappa}{m}-\omega^2\right)a_{21}=-\frac{\kappa}{m}a_{11}-\frac{\kappa}{m}a_{21}=-\frac{\kappa}{m}(a_{12}+a_{22})=0$$

Each of these two equations tells us that $a_{12} = -a_{22}$.

4. **Determine the scale factors required to match the initial conditions**. In this example, we do not need to match initial conditions (such as the initial displacement or the initial velocity and we thus do not need to determine scale factor).

5. **Determine the normal coordinates**. The normal coordinates are those coordinates that oscillate with a single frequency. In the current example we thus observe the following normal coordinates:

$$\eta_1 = a_{11}\theta_1 + a_{21}\theta_2 = a_{11}(\theta_1 + \theta_2)$$

$$\eta_2 = a_{12}\theta_1 + a_{22}\theta_2 = a_{22}(\theta_1 - \theta_2)$$

Note: the constants in these equations need to be adjusted to match the initial conditions.
The system will carry out a motion with normal frequency 1 when $\eta_2 = 0$. This requires that $\theta_1 = \theta_2$ and the motion is symmetric. The system will carry out a motion with normal frequency 2 when $\eta_1 = 0$. This requires that $\theta_1 = -\theta_2$ and the motion is asymmetric.

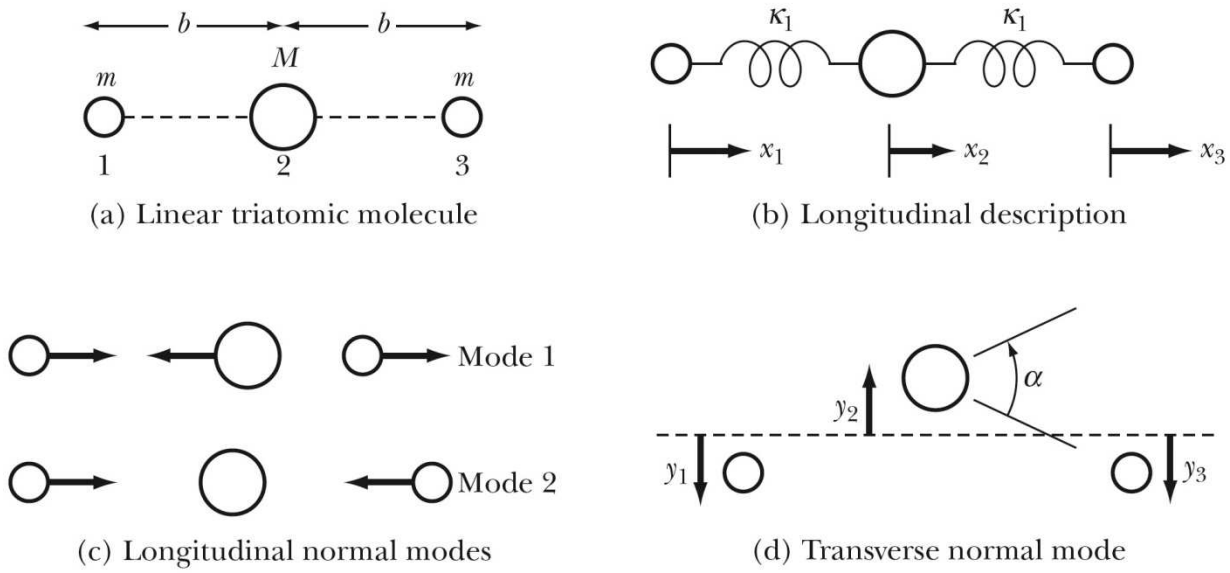
Molecular Vibrations

Our theory of coupled oscillations has many important applications in molecular physics. Each atom in a molecule has 3 degrees of freedom, and if we are looking at a molecule with n atoms, we have a total of $3n$ degrees of freedom. Three different types of motion can be carried

out by the atoms in the molecule: translation (3 degrees of freedom), rotation (3 degrees of freedom), and vibration ($3n$ - 6 degrees of freedom).

Consider a linear molecule (the equilibrium positions of all atoms are located along a straight line) with n atoms. The number of degrees of freedom associated with Vibrational motion is $3n$ – 5 since there are only 2 rotational degrees of freedom. The vibrations in a linear molecule can be longitudinal vibrations (there are n - 1 degrees of freedom associated with this type of vibrations) and transverse vibrations (there are $(3n - 5) - (n - 1) = (2n - 4)$ degrees of freedom associated with this type of vibration). If the vibrations are planar vibrations (the motion of all atoms is carried out in a single plane) we can specify any transverse vibration in terms of vibrations in two mutually perpendicular planes. The characteristic frequencies in each of these planes will be the same (symmetry) and the number of characteristic frequencies will thus be equal to n - 2.

To illustrate molecular vibrations let us consider the dynamics of a triatomic molecule (see Figure 5).



(a) Linear triatomic molecule (b) Longitudinal description

(c) Longitudinal normal modes (d) Transverse normal mode

Figure 5. Vibrational motion of a linear triatomic molecule.

In order to determine the vibrational modes of this system we look at the longitudinal and transversal modes separately. Since we are not interested in pure translational motion we can require that the center of mass of the system is at rest. This means that we do not have 3 independent position coordinates, but only 2. For example, we can eliminate the position of the heavy atom:

$$x_2 = -\frac{m}{M}(x_1 + x_3)$$

In order to determine the normal modes, we will follow the same procedure as we used in the previous example (note: this differs from the approach used in the textbook).

1. Choose generalized coordinates. The proper generalized coordinates in this problem are the displacements x_1 and x_2. The kinetic and the potential energy of the system can be easily expressed in terms of these displacements. The kinetic energy of the system is thus just equal to the kinetic energy of the three atoms, and thus equal to

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 = \\ &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\left(\frac{m}{M}\right)^2\left(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_3 + \dot{x}_3^2\right) + \frac{1}{2}m\dot{x}_3^2 = \\ &= \frac{1}{2}\left(m + \frac{m^2}{M}\right)\dot{x}_1^2 + \left(\frac{m^2}{M}\right)\dot{x}_1\dot{x}_3 + \frac{1}{2}\left(m + \frac{m^2}{M}\right)\dot{x}_3^2 \end{aligned}$$

The potential energy of the system is the sum of the potential energy associated with the compression of the springs. The total potential energy is thus equal to

$$\begin{aligned} U &= \frac{1}{2}\kappa_1(x_2 - x_1)^2 + \frac{1}{2}\kappa_1(x_3 - x_2)^2 = \\ &= \frac{1}{2}\kappa_1\left\{\left(x_1^2 + x_3^2\right) + 2x_2^2 - 2x_2(x_1 + x_3)\right\} = \\ &= \frac{1}{2}\kappa_1\left\{\left(x_1^2 + x_3^2\right) + 2\frac{m^2}{M^2}(x_1 + x_3)^2 + 2\frac{m}{M}(x_1 + x_3)^2\right\} = \\ &= \frac{1}{2}\kappa_1\left\{\left(1 + 2\frac{m}{M} + 2\frac{m^2}{M^2}\right)\left(x_1^2 + x_3^2\right) + 2\left(2\frac{m}{M} + 2\frac{m^2}{M^2}\right)x_1x_3\right\} \end{aligned}$$

2. Determine the *A* and *m* tensors. In order to calculate these tensors we use the expressions for T and U obtained in step 1. Since the kinetic energy obtained in step 1 does not contain products of the generalized velocity of mass 1 and the generalized velocity of mass 2, the mass tensor will be a diagonal tensor. We can see this by looking at the definition of the mass tensor elements:

$$T = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k$$

The mass tensor is thus equal to

$$m=\begin{Bmatrix}\left(m+\frac{m^2}{M}\right) & \left(\frac{m^2}{M}\right)\\ \left(\frac{m^2}{M}\right) & \left(m+\frac{m^2}{M}\right)\end{Bmatrix}$$

The A tensor is equal to

$$A=\begin{Bmatrix}\left.\frac{\partial^2 U}{\partial x_1\partial x_1}\right|_0 & \left.\frac{\partial^2 U}{\partial x_1\partial x_3}\right|_0\\ \left.\frac{\partial^2 U}{\partial x_3\partial x_1}\right|_0 & \left.\frac{\partial^2 U}{\partial x_3\partial x_3}\right|_0\end{Bmatrix}=\begin{Bmatrix}\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right) & 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)\\ 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right) & \kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)\end{Bmatrix}$$

3. Determine the eigen frequency and the eigen vectors. The eigen frequencies can be determined by requiring that the determinant of the coefficients of the equations of motions vanishes:

$$\left|\{A\}-\{m\}\omega^2\right| = \left|\begin{Bmatrix}\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right) & 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)\\ 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right) & \kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)\end{Bmatrix}-\begin{Bmatrix}\left(m+\frac{m^2}{M}\right) & \left(\frac{m^2}{M}\right)\\ \left(\frac{m^2}{M}\right) & \left(m+\frac{m^2}{M}\right)\end{Bmatrix}\omega^2\right| =$$

$$= \begin{vmatrix}\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2 & 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\\ 2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2 & \kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\end{vmatrix}=0$$

This requires that

$$\left(\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right)^2-\left(2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right)^2=0$$

or

$$\left(\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right)=\pm\left(2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right)$$

Consider the two signs. First the positive sign:

$$\left(\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right)=\left(2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right)$$

This is equivalent to

$$m\omega^2=\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)=\kappa_1$$

or

$$\omega=\sqrt{\frac{\kappa_1}{m}}$$

Now consider the negative sign:

$$\left(\kappa_2\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right)=-\left(2\kappa_2\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right)$$

This is equivalent to

$$m\left(m+2\frac{m}{M}\right)\omega^2=\kappa_2\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)+2\kappa_2\frac{m}{M}\left(1+\frac{m}{M}\right)=\kappa_2\left(1+4\frac{m}{M}\left(1+\frac{m}{M}\right)\right)=\kappa_2\left(1+2\frac{m}{M}\right)^2$$

or

$$\omega=\sqrt{\frac{\kappa_2\left(1+2\frac{m}{M}\right)^2}{m\left(1+2\frac{m}{M}\right)}}=\sqrt{\frac{\kappa_2}{m}\left(1+2\frac{m}{M}\right)}=\sqrt{\frac{\kappa_2}{m}\left(1+2\frac{m}{M}\right)}=\sqrt{\kappa_2\left(\frac{M+2m}{mM}\right)}$$

Consider the first eigen frequency, and assume the corresponding eigen vector is (a_{11}, a_{21}). The equations of motion for this frequency are

$$\left\{\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right\}a_{11}+\left\{2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right\}a_{21}=0$$

$$\left\{2\kappa_1\frac{m}{M}\left(1+\frac{m}{M}\right)-\left(\frac{m^2}{M}\right)\omega^2\right\}a_{11}+\left\{\kappa_1\left(1+2\frac{m}{M}\left(1+\frac{m}{M}\right)\right)-\left(m+\frac{m^2}{M}\right)\omega^2\right\}a_{21}=0$$

Substituting the expression of the first eigen frequency in these equations we obtain for each equation the following expression:

$$\left\{\kappa_1\left(\frac{m}{M}\left(1+2\frac{m}{M}\right)\right)\right\}a_{11}+\left\{\kappa_1\frac{m}{M}\left(1+2\frac{m}{M}\right)\right\}a_{31}=\kappa_1\left(\frac{m}{M}\left(1+2\frac{m}{M}\right)\right)(a_{11}+a_{31})=0$$

This equations tells us that $a_{11} = -a_{31}$. Since the eigen vectors are orthogonal, we expect that the eigen vector for the second eigen frequency is given by $a_{12} = a_{32}$.

4. Determine the scale factors required to match the initial conditions. In this example, we do not need to match initial conditions (such as the initial displacement or the initial velocity and we thus do not need to determine scale factor).

5. Determine the normal coordinates. The normal coordinates are those coordinates that oscillate with a single frequency. In the current example we thus observe the following normal coordinates:

$$\eta_1 = a_{11}x_1 + a_{31}x_3 = a_{11}(x_1 - x_3)$$

$$\eta_2 = a_{12}x_1 + a_{32}x_3 = a_{22}(x_1 + x_3)$$

Note: the constants in these equations need to be adjusted to match the initial conditions.
The system will carry out a motion with normal frequency 1 when $\eta_2 = 0$. This requires that $x_1 = -x_3$ and the motion is asymmetric. The system will carry out a motion with normal frequency 2 when $\eta_1 = 0$. This requires that $x_1 = x_3$ and the motion is symmetric.
Note: the normal frequency 1 is equal to the frequency of a mass m on a spring whose other end remains fixed. This mode requires the central atom to remain fixed, and this can be achieved when the motion is asymmetric since the forces exerted by the two springs on the central mass cancel.

The transverse vibration of the molecule can be specified in terms of a single parameter α. For this mode of vibration we will get a single "uncoupled" differential equation with a single corresponding characteristic frequency. The calculation of this frequency is shown in detail in the text book and will not be reproduced here.

The Loaded String

A good model of an elastic string is a string of particles of mass m, each separated by a distance d (see Figures 6 and 7). We will assume that the tension in the string is constant and equal to τ.

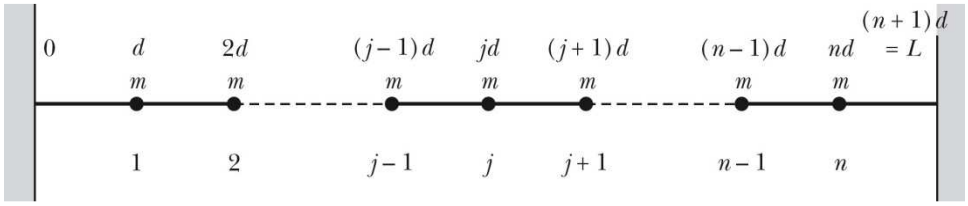


Figure 6. The loaded string.

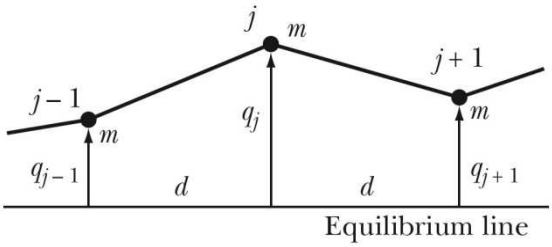


Figure 7. Calculation of the restoring force acting on mass *j*.

In examining this problem, we will make the following assumptions:

- • The masses can only move in the vertical direction (thus only the component of the tension in the vertical direction matters).
- • The potential energy of the system is the potential energy associated with the tension in the string.
- • We assume that the displacements from the equilibrium positions are small.
- • We ignore the gravitational forces acting on the masses (and the associated gravitational potential energy).

In order to calculate the force acting on mass *j* we calculate the vertical components due to the tension in the left and right section of the string:

$$F_i = -\tau\left(\frac{q_i - q_{i-1}}{\sqrt{d^2 - (q_i - q_{i-1})^2}}\right) - \tau\left(\frac{q_i - q_{i+1}}{\sqrt{d^2 - (q_i - q_{i+1})^2}}\right) \approx \frac{\tau}{d}\left(q_{i-1} - 2q_i + q_{i+1}\right)$$

In the last step we have made the assumption that the vertical displacement is small compared to the distance *d*. Since the force on mass *j* depends not only on the position of mass *j* but also on the position of masses *j* - 1 and *j* + 1. We can use the force on the *n* masses to obtain *n* coupled differential equations that we can try to solve. Consider the following trial function:

$$q_i(t) = a_i e^{i\omega t}$$

Substituting this function into our differential equation we obtain

$$F_i = m\ddot{q}_i = -m\omega^2 a_i e^{i\omega t} = \frac{\tau}{d}\left(a_{i-1}e^{i\omega t} - 2a_i e^{i\omega t} + a_{i+1}e^{i\omega t}\right)$$

or

$$\frac{\tau}{d}a_{j-1} - \left(2\frac{\tau}{d} - m\omega^2\right)a_j + \frac{\tau}{d}a_{j+1} = 0$$

The amplitudes a can be complex. Based on the type of motion we expect the system to carry out, we can try to parameterize the amplitude dependence on j in the following way:

$$a_j = ae^{i(j\gamma - \delta)}$$

where a is now a real number. Taking this expression for a_j and substituting it into the previous equation we obtain

$$ae^{-i\delta}\left(\frac{\tau}{d}e^{-i\gamma} - \left(2\frac{\tau}{d} - m\omega^2\right) + \frac{\tau}{d}e^{i\gamma}\right) = 0$$

This expression can be used to find the following expression for the angular frequency:

$$m\omega^2 = 2\frac{\tau}{d} - \frac{\tau}{d}\left(e^{-i\gamma} + e^{i\gamma}\right) = 2\frac{\tau}{d} - 2\frac{\tau}{d}\cos\gamma = 2\frac{\tau}{d}(1 - \cos\gamma) = 4\frac{\tau}{d}\sin^2\frac{\gamma}{2}$$

or

$$\omega = 2\sqrt{\frac{\tau}{md}}\sin\frac{\gamma}{2}$$

Since there must be n eigen frequencies, we expect to find n distinct values of γ.

Additional constraints are imposed on the solution by requiring that the boundary conditions are met:

- • $a_0 = 0$: This condition requires that (note: we only consider the real part of the amplitude)

  $$a_0 = a\cos(0\gamma - \delta) = a\cos(\delta) = 0$$

  or

  $$\delta = \frac{\pi}{2}$$

- • $a_{n+1} = 0$: This condition requires that

$$a_{n+1} = a\cos\left((n+1)\gamma - \frac{\pi}{2}\right) = a\sin\left((n+1)\gamma\right) = 0$$

The argument of the sin function must thus be an integer multiple of π:

$$(n+1)\gamma_s = s\pi$$

or

$$\gamma_s = \frac{s\pi}{(n+1)}$$

where $s = 1, 2, 3, \ldots, n$.

Since the boundary conditions provide us with n different values of the parameter γ, we expect that there will also be n unique values of the angular frequency for this system:

$$\omega_s = 2\sqrt{\frac{\tau}{md}}\sin\left(\frac{s\pi}{2(n+1)}\right)$$

where $s = 1, 2, 3, \ldots, n$.

Putting all the different pieces of information together we can now write down the general solution of the loaded string problem:

$$q_{j,s}(t) = a_{j,s}e^{i\omega_s t} = a_s \sin\left(j\frac{s\pi}{(n+1)}\right)e^{i\sqrt{\frac{\tau}{md}}\left(\frac{s\pi}{2(n+1)}\right)}$$

and

$$q_j(t) = \sum_s \beta_s a_s \sin\left(j\frac{s\pi}{(n+1)}\right)e^{i\sqrt{\frac{\tau}{md}}\left(\frac{s\pi}{2(n+1)}\right)}$$

We can also use the Lagrangian method to find the normal modes of the system, but as we will see, this approach is much less transparent than the approach just used. In order to apply this procedure we need to determine the kinetic energy and the potential of the system in terms of the generalized coordinates. In this particular problem, the best choice for the generalized coordinates is the vertical displacement of the masses. In terms of these displacements we can write the kinetic energy as

$$T = \frac{1}{2} m \sum_{i=1}^{n} \dot{q}_i^2$$

In order to determine the potential energy of the system, we first have to determine the potential energy of mass *j*. Since we know the relation between the potential energy and the force, we can see that the potential energy is equal to

$$U = \frac{\tau}{2d} \sum_{i=1}^{n+1} (q_{i-1} - q_i)^2$$

Note: the index runs from $j = 1$ to $j = n + 1$. There are no masses at position 0 and at position $(n+1)d$; these positions are the ends of the string. The displacement at these locations is equal to 0.

Note: in order to verify that the potential energy is correct, we need to show that its gradient is related to the force on mass *j*:

$$F_i = -\frac{\partial U}{\partial q_i} = -\frac{\tau}{2d} \frac{\partial}{\partial q_i} \left\{ (q_{i-1} - q_i)^2 + (q_i - q_{i+1})^2 \right\} = -\frac{\tau}{d} \left\{ -(q_{i-1} - q_i) + (q_i - q_{i+1}) \right\} = \frac{\tau}{d} \left\{ q_{i-1} - 2q_i + q_{i+1} \right\}$$

The mass tensor *m* for the system is given by

$$\{m\} = \begin{pmatrix} m & 0 & 0 & \cdots \\ 0 & m & 0 & \cdots \\ 0 & 0 & m & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The potential tensor *A* for the system is given by

$$\{A\} = \begin{pmatrix} \frac{\partial^2 U}{\partial q_1 \partial q_1} & \frac{\partial^2 U}{\partial q_1 \partial q_2} & \frac{\partial^2 U}{\partial q_1 \partial q_3} & \cdots \\ \frac{\partial^2 U}{\partial q_2 \partial q_1} & \frac{\partial^2 U}{\partial q_2 \partial q_2} & \frac{\partial^2 U}{\partial q_2 \partial q_3} & \cdots \\ \frac{\partial^2 U}{\partial q_3 \partial q_1} & \frac{\partial^2 U}{\partial q_3 \partial q_2} & \frac{\partial^2 U}{\partial q_3 \partial q_3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \frac{2\tau}{d} & -\frac{\tau}{d} & 0 & \cdots \\ -\frac{\tau}{d} & \frac{2\tau}{d} & -\frac{\tau}{d} & \cdots \\ 0 & -\frac{\tau}{d} & \frac{2\tau}{d} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The eigen frequencies can now be found by requiring that the secular determinant is equal to 0:

$$\begin{vmatrix} \frac{2\tau}{d} - m\omega^2 & -\frac{\tau}{d} & 0 & \cdots \\ -\frac{\tau}{d} & \frac{2\tau}{d} - m\omega^2 & -\frac{\tau}{d} & \cdots \\ 0 & -\frac{\tau}{d} & \frac{2\tau}{d} - m\omega^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

We can solve this equation for ω but the results are more difficult to interpret than the results obtained with out first approach.